Guess the Problem!

June 29th, 2020 to July 15th, 2020

Background: With the OTJMO, USOJMO, and USOMO, three very advanced and highly challenging competitions, having finished, we feel like there should be a separate "mini-event"-type competition going on that is more approachable for all skill levels. Thus, we, the OTSS committee, have decided to release our first mini-event: **Guess the Problem!** In this mini-event, you will be given six "problems". These "problems" are actually the solutions for six of our OTSS problems left unused. Your goal is to guess the problem statement that corresponds to the solution based on certain hints, such as wording, names, computations, etc. Each problem will be worth seven points, for a total of 42 points. Each problem will be scored as follows:

1. Closeness (4 points max.): Obviously, your problem statement will not look exactly like our problem (unless you're secretly part of our committee), but if your problem does not match up with the solution, we will have to deduct points in this category. Note that this has little to do with how much the problem matches what we have; this is about how much the problem fits the solution.

2. Style (2 points max.): A very important part of a problem is how it sounds. This means that your problem should not have stylistic issues nor should it be overly verbose nor succinct. It should also be similar to the flavor text of the actual problem. For example, if you write a story about Alice and Bob being two points with a string that connects them instead of saying $\overline{AB}$, you will lose points. *But if you want to make intentionally off-flavor problems, feel free to do so! (Just make sure they are appropriate.) A suggestion for that is to do it only after you take the test for real, and then submit the other version as an unofficial copy which will not be graded. Make sure you identify which copy is the one we will grade.* Note that if you submit a problem statement that is stylistically pleasing but does not earn any of the points above, you will not earn any points in this category either.

3. Mechanics (1 point max.): If your problem has obvious and consistent spelling or grammatical errors, we will have to deduct one point. Note that if you submit a problem statement that is mechanically correct but does not earn any of the points above, you will not earn a point in this category either.

4. Optional (0 points max., but some other special reward will be provided): If you correctly guess the username of the person who wrote the solution to one or more of the problems, you will be rewarded in some way that we have yet to decide. Note that you do not have to answer those respective problem(s) to earn this reward.

Instructions: We recommend you type up the problems using a LaTeX editor or Google Docs, and save your problems file as a PDF. Alternatively, you can screenshot your work. Then, you can private message the OTSS Committee member DeToasty3 on Art of Problem Solving or Discord with your PDF or image file. They will grade your problems and give you your score within 24 hours after you submit. A final leaderboard will be posted a few days after July 15th, 2020. If you wish to keep your AoPS username anonymous should you make the leaderboard, please include this information in your private message. You will have 75 minutes to work on these six problems.

"Problems"

1. The value of a base 8 number consisting of only ones converted to base 10 equals $\frac{8^n-1}{7}$ where n is a nonnegative integer. Factoring the numerator results in $\frac{(2^n-1)(4^n+2^n+1)}{7}$. Take the expression $\pmod 3$ on the numerator to prove that $7 \mid 2^n - 1$ when $n \equiv 0 \pmod 3$ and $7 \mid 4^n + 2^n + 1$ otherwise. Now, it is easy to see that $\frac{(2^n-1)(4^n+2^n+1)}{7}$ will be composite unless $n = 0, 1, 3$. Testing, only $n = \boxed{3}$ is prime.

2. Note that the smallest value of N that gives a 3-digit base-three representation and a 2-digit base-four representation is 9, and the largest value of N is 15. We can test all values of N such that $d \neq e$ and $9 \leq N \leq 15$: $21_4 = 9$, $23_4 = 11$, $30_4 = 12$, $31_4 = 13$, and $32_4 = 14$. We then get that $9 = 100_3$, $11 = 102_3$, $12 = 110_3$, $13 = 111_3$, and $14 = 112_3$. The only value of N that satisfies the condition that a, b, and c are all distinct is $\boxed{011}$.

3. The total amount of sheets of paper is $\frac{n(n+1)}{2}$, so the expected value is
$$1\left(\frac{1}{\frac{n(n+1)}{2}}\right) + 2\left(\frac{2}{\frac{n(n+1)}{2}}\right) + \cdots + n\left(\frac{n}{\frac{n(n+1)}{2}}\right) = \frac{1^2 + 2^2 + \cdots + n^2}{\frac{n(n+1)}{2}}.$$
This is then equivalent to
$$\frac{\frac{n(n+1)(2n+1)}{6}}{\frac{n(n+1)}{2}} = \frac{2n+1}{3}.$$
This is only an integer when $n \equiv 1 \pmod 3$, so the set of remainders upon division by 3 is $\boxed{\{1\}}$.

4. We can find that the central angle is $180° - 2(90° - 37°) = 74°$, so we have to find the smallest value of x such that $74x$ is an integer multiple of 360. This is just $\frac{360°}{\gcd(74,360)} = 180°$, so our answer is $\boxed{180}$.

5. Just a load of symmetry: let O be the center of the rectangle, and clearly $BO = \sqrt{15^2 + 30^2} = 15\sqrt{5}$. By PoP, $BE \cdot 15\sqrt{5} = 15^2$ so $BE = \frac{15}{\sqrt{5}} = 3\sqrt{5}$ so $EO = 12\sqrt{5}$ and we get two of 'em so it's $24\sqrt{5}$ so the answer is $\boxed{029}$.

6. Let each of the five segments have lengths a, b, c, d, and e, in order from left to right. Without loss of generality, let their sum be 1. In order for the conditions to not be met, we have to have the largest segment be greater than or equal to $\frac{1}{2}$ by exploiting the Triangle Inequality with five lengths. Consider the midpoint of the whole line segment before it is cut. If the longest segment is greater than $\frac{1}{2}$, then if we order the five line segments from shortest to longest, all four of the cuts will be to the left side of the midpoint. Then, for each of the four cuts, there is a $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$ chance that they will all end up on the left side of the midpoint, with a $\frac{1}{4!} = \frac{1}{24}$ chance that the four shorter line segments will already be in order from shortest to longest, for a total probability of $\frac{1}{16 \cdot 24}$. Now, since there are $5! = 120$ ways to assign each of a, b, c, d, and e with one of the five line segment lengths, our overall probability is $120 \cdot \frac{1}{16 \cdot 24} = \frac{5}{16}$. Note that even if there were two or more segments with the same length, this probability would not change. Since we found the probability of the complement, our actual answer is $1 - \frac{5}{16} = \frac{11}{16}$, so $a + b = \boxed{027}$.

Time: 75 minutes.
Each problem is worth 7 points.