OTSS TMC
Spring Mathematics Competitions

Spring Mathematics Competitions

1st Annual

OTIE I

Olympiad Test Invitational Examination I

Tuesday, May 26, 2020

General Information/Guidelines

1. DO NOT OPEN THIS BOOKLET UNTIL YOU GIVE THE SIGNAL TO BEGIN.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers. There is neither partial credit nor a penalty for wrong answers.
3. No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, **calculators, calculating devices, smart phones or watches, and computers are not permitted.**
4. Unlike the American Mathematics Competitions, a combination of the OTIE and the TMC 10/12 scores are not used to determine eligibility for participation in the Olympiad Test Junior Mathematical Olympiad (OTJMO). In particular, anyone can participate in the OTJMO, which will be given from Saturday, May 30, 2020, to Tuesday, June 16, 2020.
5. Record all your answers, but not identification information, on the OTIE answer form. Only the answer form will be collected from you.

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1. Xu and Zhao can each solve m puzzles in an hour, and Wei can solve n puzzles in an hour, where m and n are positive integers. Xu starts working on a set of $m \cdot n$ puzzles for 3 hours. Then, Xu, Zhao, and Wei all work for the next 5 hours. Finally, Wei works on the set alone for 1 more hour and finishes the set. Find the greatest possible value of $m + n$.

2. Jela and Benn are playing a game. Each round, Jela and Benn each flip a fair coin at the same time. Jela and Benn win if they flip heads together. However, they lose if they flip tails together for three rounds in a row. If neither event happens after the end of 4 rounds, they also lose. The probability that Jela and Benn win can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

3. Two dogs, Otie and Amy, are each given an integer number of biscuits to eat, where Otie and Amy get x and y biscuits, respectively, and $0 < x < y < 72$. At the start, the numbers x, y, and 72 form an arithmetic progression, in that order. Each dog then eats N of their biscuits, where N is a positive integer less than x. After they finish eating, Amy now has exactly three times the number of biscuits left over as Otie. Find the number of possible values of N.

4. April and Ollie have 10 empty baskets. In each basket, April puts a whole number of flowers between 1 and 11 inclusive, chosen uniformly and randomly. Then, Ollie puts a whole number of flowers in each basket between 1 and 12 inclusive, chosen uniformly and randomly. Finally, April and Ollie compute the product of the number of flowers in each basket over all 10 baskets. Given that the expected value of the product is $\frac{m}{n}$ for relatively prime positive integers m and n, find the remainder when $m + n$ is divided by 1000.

5. Let rectangle $ABCD$ have $AB = 24$ and $BC = 10$. A point P on $\overline{AB}$ is chosen uniformly at random. The probability there exists a point Q on $\overline{AD}$ such that P is on the perpendicular bisector of segment CQ is $\frac{m}{n}$ for relatively prime positive integers m and n. Find $m + n$.

6. A positive integer is called *maybe prime* if all of its digits are primes and the number is not divisible by 2 or 3. Find the number of positive integers less than $10,000$ that are *maybe prime*.

7. In acute $\triangle ABC$, altitudes $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ intersect at point H. Line AH intersects the circumcircle of $\triangle BCH$ at another point A', where $A' \neq H$. Given that $A'D - AH = 3$, $BD = 4$, and $CD = 6$, find the area of quadrilateral $AEHF$.

8. A 7×7 square chessboard has gridlines parallel to the edges of the board which split it into 49 congruent 1×1 squares. Jacob wants to cover this chessboard with four indistinguishable 3×3 square tiles such that none of the tiles overlap or go off the edge of the board, and all of the sides of each tile are perfectly aligned with the gridlines of the board. How many ways are there to tile the grid such that at least one tile touches a corner of the board? Two tilings are considered distinct if they are not identical without rotating or reflecting the chessboard.

9. On each vertex of regular hexagon $ABCDEF$, where the vertices are distinct, a positive integer divisor of 2020 is written. Then, on each edge, the greatest common divisor of the two integers on the vertices containing the edge is written. Suppose the least common multiple of the six integers written on the edges is 2020. If N is the number of ways where this is possible, find the sum of the (not necessarily distinct) primes in the prime factorization of N.

10. It is given that the equation $x^3 + 2x^2 + 4x + 9 = 0$ has a unique real solution x such that
$$\lfloor 10^{11} x \rfloor = -211,785,097,233.$$
Find the sum of the digits of $\lfloor 10^{11}(x^4+4) \rfloor$. Note that $\lfloor r \rfloor$ denotes the greatest integer less than or equal to r for all real numbers r.

11. Let ω be the incircle of $\triangle ABC$ and denote D and E as the tangency points of ω with sides $\overline{BC}$ and $\overline{AB}$, respectively. Line AD intersects ω at two distinct points, D and F. The circle passing through E that is tangent to line AD at F intersects line AB at two distinct points, E and G. Given that $AG < AE$, $AG = 24$, $EF = 20$, and $DE = 25$, length BE can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find the remainder when $m + n$ is divided by 1000.

12. The value of
$$\frac{\sin^9\left(\frac{\pi}{18}\right) - 1}{5\sin\left(\frac{\pi}{18}\right) - 3\sin^2\left(\frac{\pi}{18}\right) - 15}$$
can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Find $m + n$.

13. Let $\triangle ABC$ be an acute triangle with $BC = 2AC$. Let D be the midpoint of $\overline{BC}$ and E be the foot of the perpendicular from B to $\overline{AC}$. Lines BE and AD intersect at F such that $AF = 2CE$. The degree measure of angle C can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

14. Define the function
$$S(n) = \sum_{k=1}^{n} \left(k \left\lfloor \frac{n}{k} \right\rfloor\right)$$
for all positive integers n, where $\lfloor r \rfloor$ denotes the greatest integer less than or equal to r for all real numbers r. Find the sum of all positive integers n such that
$$S(2n) + S(n-1) - S(2n-1) - S(n) = 48.$$

15. In cyclic quadrilateral $ABCD$, $\overline{CD}$ is extended past C to intersect line AB at B', and $\overline{AD}$ is extended past D to intersect line BC at point D'. The circumcircles of $\triangle BB'C$ and $\triangle DD'C$ intersect at another point C', where $C' \neq C$. Given that $B'C' = 12$, $B'B = D'C' = 8$, and $D'D = 4$, length AC' can be expressed as $a\sqrt{b}$, where a and b are positive integers and b is not divisible by the square of any prime. Find $a + b$.

The 2020 OTIE Solution Pamphlet will be released after the testing period.

Olympiad Test Spring Series

CONTACT US

Correspondence about the problems and solutions for this OTIE, orders for any of our publications, or any other queries may be addressed to: otss.contactus@gmail.com.

Alternatively, if you are a member of Art of Problem Solving*, then you can also send a Private Message to* **Emathmaster** & **kevinmathz** *with your queries.*

2020 OTJMO

The Olympiad Test Junior Mathematical Olympiad (OTJMO) is a **6**-question, 9-hour, essay-type examination. The best way to prepare for the OTJMO is to study previous years of the exam. Copies need not be ordered from anywhere, and will only be available once released by OTSS.

PUBLICATIONS For a complete listing of our publications, please visit our website.

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Finally, we thank you for taking this mock. We hope you enjoyed it!