2020 OTIE I Problems and Solutions Document

Olympiad Test Spring Series

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1. Xu and Zhao can each solve m puzzles in an hour, and Wei can solve n puzzles in an hour, where m and n are positive integers. Xu starts working on a set of $m \cdot n$ puzzles for 3 hours. Then, Xu, Zhao, and Wei all work for the next 5 hours. Finally, Wei works on the set alone for 1 more hour and finishes the set. Find the greatest possible value of $m + n$.

   Proposed by Emathmaster

   (Answer: 098)

   The total number of hours put in by Xu and Zhao combined is $3 + 5 \cdot 2 = 13$, and Wei puts in 6. Thus, $mn = 13m + 6n$. Rearranging, we get $mn - 13m - 6n = 0$, and adding 78 to both sides and factoring the left hand side, we get $(m - 6)(n - 13) = 78$. In order to maximize $m + n$, we have to make $(m - 6)$ and $(n - 13)$ as far as possible, which happens when either $m - 6 = 78$ and $n - 13 = 1$, or when $m - 6 = 1$ and $n - 13 = 78$. In both cases, we get that $m + n = 78 + 1 + 6 + 13 = \boxed{098}$.

2. Jela and Benn are playing a game. Each round, Jela and Benn each flip a fair coin at the same time. Jela and Benn win if they flip heads together. However, they lose if they flip tails together for three rounds in a row. If neither event happens after the end of 4 rounds, they also lose. The probability that Jela and Benn win can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

   Proposed by ivyzheng and DeToasty3

   (Answer: 215)

   Consider the complement; where either three tails-tails pairs occur in a row, or 4 rounds happen with no event happening. In this complementary case, we can have any of the three possibilities H-T, T-H, or T-T, where none of the turns are H-H. (where H represents heads, T represents tails, the first letter in the pair represents Jela's flip, and the second letter in the pair represents Benn's flip) There are 4 rounds, and there are three possibilities for each turn, for a total of $3^4 = 81$ ways. However, there is also the one case T-T T-T T-T H-H, which contributes 1 more case to our count, for a total of 82 ways.

   Now, the probability is $1 - \frac{82}{256} = \frac{256-82}{256} = \frac{174}{256} = \frac{87}{128}$, so $m + n = 87 + 128 = \boxed{215}$.

3. Two dogs, Otie and Amy, are each given an integer number of biscuits to eat, where Otie and Amy get x and y biscuits, respectively, and $0 < x < y < 72$. At the start, the numbers x, y, and 72 form an arithmetic progression, in that order. Each dog then eats N of their biscuits, where N is a positive integer less than x. After they finish eating, Amy now has exactly three times the number of biscuits left over as Otie. Find the number of possible values of N.

   Proposed by DeToasty3

   (Answer: 014)

   Let d be the common difference of the arithmetic progression. Then Otie has $72 - 2d$ pieces, and Amy has $72 - d$ pieces. After eating, Otie has $72 - 2d - N$ pieces, and Amy has $72 - d - N$ pieces. We set up a system:

$$216 - 6d - 3N = 72 - d - N$$

$$N < 72 - 2d$$

$$2d < 72.$$

   Simplifying, we get $5d + 2N = 144$, $4d + 2N < 144$, and $d < 36$. Subtracting $5d + 2N = 144$ and $4d+2N < 144$, we get that $d > 0$, which is always true. Therefore, any integer solution to the equation $5d+2N = 144$, where d is a positive integer, will necessarily satisfy the inequality $4d+2N < 144$. We don't want $N = 0$ because N is defined as a positive integer, so all integer solutions to the equation $5d+2N = 144$, where d and N are both positive integers, satisfy our conditions. The integer solution with the smallest positive value of N is $(28, 2)$. The integer solution with the largest integer value of N is $(2, 67)$. ($N \neq 72$ because if $N = 72$, then $d = 0$, which we don't want.) We can repeatedly change N by -5 and change d by 2 to get other solutions. We see that in all of these solutions, $d < 36$. Thus, our answer is $\frac{67-2}{5} + 1 = \boxed{014}$.

4. April and Ollie have 10 empty baskets. In each basket, April puts a whole number of flowers between 1 and 11 inclusive, chosen uniformly and randomly. Then, Ollie puts a whole number of flowers in each basket between 1 and 12 inclusive, chosen uniformly and randomly. Finally, April and Ollie compute the product of the number of flowers in each basket over all 10 baskets. Given that the expected value of the product is $\frac{m}{n}$ for relatively prime positive integers m and n, find the remainder when $m + n$ is divided by 1000.

   Proposed by Emathmaster

   (Answer: 649)

   The expected value on the first distribution of flowers in any given basket is 6. The expected value of flowers between 1 to 12 is $\frac{13}{2}$, so we expect $\frac{25}{2}$ flowers in each of the ten baskets at the end. By Linearity of Expectation, we have that the expected value of the product is $\left(\frac{25}{2}\right)^{10}$. Then, $m + n = 25^{10} + 2^{10}$. Note that $25^{10} \equiv 625 \pmod{1000}$ and $2^{10} = 1024$. So the remainder when $m + n$ is divided by 1000 is $625 + 24 = \boxed{649}$.

5. Let rectangle $ABCD$ have $AB = 24$ and $BC = 10$. A point P on $\overline{AB}$ is chosen uniformly at random. The probability there exists a point Q on $\overline{AD}$ such that P is on the perpendicular bisector of segment CQ is $\frac{m}{n}$ for relatively prime positive integers m and n. Find $m + n$.

   Proposed by Emathmaster

(Answer: 313)

Let O be the center of the rectangle. Let M be the midpoint of side $\overline{AB}$ and N be the point such that $\overline{NO}$ is perpendicular to $\overline{AC}$. Then, P is bounded by M and N, which gives the answer of $\frac{25}{288}$. Then, $m + n = \boxed{313}$.

6. A positive integer is called *maybe prime* if all of its digits are primes and the number is not divisible by 2 or 3. Find the number of positive integers less than 10,000 that are *maybe prime*.

   Proposed by PCChess

   (Answer: 170)

   Solution 1: Note that we can only use the digits $2, 3, 5$ and 7. Consider an n digit number. For the first $n - 1$ digits (all the digits except the units digit), there are 4^{n-1} ways to choose the digits so that the number is maybe prime. Since a number is divisible by 3 if and only if its digits sum to a multiple of 3, and the fact that $3, 5$ and 7 are all different $\mod 3$, two of $3, 5$ and 7 will be able to be the units digit of the n digit maybe prime number. Hence, for any n, there are $2 \cdot 4^{n-1}$ maybe prime numbers. The answer is therefore $2 + 8 + 32 + 128 = \boxed{170}$.

   Solution 2: First, notice that we can only use the digits $2, 3, 5,$ and 7. Each of those digits are $2, 0, 2,$ and $1 \mod 3$, respectively. We can do casework on the number of digits.

   1-digit numbers: There are 2 numbers: 5 and 7.

   2-digit numbers: There are so little that we can list them out: $23, 25, 35, 37, 53, 55, 73, 77$. We have 8 numbers.

   3-digit numbers: Note that there are $4^3 = 64$ total possibilities. We can now do complementary counting and PIE. There are 4^2 numbers divisible by 2.

   For a number to be divisible by 3, the sum of the digits have to be $0 \mod 3$. The possible residue combinations are $000, 111, 120,$ and 222. For 000, there is only one way, namely the number 333. For 111, there is only one way as well, namely 777. For 120, there are the possibilities of 237 and 357, with 6 permutations each, so there are 12 possible numbers. For 222, there are 8 possible numbers: 2 choices for each digit.

   Also, there are 6 numbers that are divisible by 2 and 3: 552, 522, 252, 222, 372, and 732. Therefore, there are $64 - (16 + 22 - 6) = 32$ 3 digit numbers that are maybe prime.

   4-digit numbers: There are $4^4 = 256$ possibilities, with $4^3 = 64$ of them divisible by 2. The possible residue combinations to be divisible by 3 are $0000, 1110, 1200, 2220$ and 2211. For 0000, there is one way: 3333. For 1110, there are 4 ways: the permutations of 7773. For 1200, there are 12+12=24 ways: the permutations of 7533 and 7233. For 2220, there are 4+12+12+4=32 ways: A 3 and the possible combinations of 5 and 2. For 2211, there are 6+12+6=24 ways: 2 sevens and the possible combinations of 5 and 2.

   There are $3+(3+6+3)+(3+3) = 21$ multiples of 2 and 3, so there are $256-(64+85-21) = 128$ maybe prime numbers.

   In total, the answer is $2 + 8 + 32 + 128 = \boxed{170}$.

7. In acute $\triangle ABC$, altitudes $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ intersect at point H. Line AH intersects the circumcircle of $\triangle BCH$ at another point A', where $A' \neq H$. Given that $A'D - AH = 3$, $BD = 4$, and $CD = 6$, find the area of quadrilateral $AEHF$.

   Proposed by Awesome_guy

(Answer: 011)

First, we note that $\angle DAC = \angle EBC = \angle HBC = \angle HA'C$. It then follows that $\triangle AA'C$ is isosceles with $AC = A'C$ and $AD = A'D$. Using $A'D - AH = 3$, $HD = 3$. Then by Power of a Point, $A'D = AD = 8$. Then, we chase similar triangles to find $HE = 3$ and $AE = 4$, so $[AHE] = 6$. Finally, $FH = \sqrt{5}$ and $AF = 2\sqrt{5}$, so $[AFH] = 5$. $[AEFH] = [AFH] + [AHE] = \boxed{011}$.

8. A 7×7 square chessboard has gridlines parallel to the edges of the board which split it into 49 congruent 1×1 squares. Jacob wants to cover this chessboard with four indistinguishable 3×3 square tiles such that none of the tiles overlap or go off the edge of the board, and all of the sides of each tile are perfectly aligned with the gridlines of the board. How many ways are there to tile the grid such that at least one tile touches a corner of the board? Two tilings are considered distinct if they are not identical without rotating or reflecting the chessboard.

   Proposed by PCChess

   (Answer: 077)

   We will do casework based on how many squares are situated in corners and where they are.

   Case 1: All 4 are in a corner.

   There is just 1 case.

   Case 2: 3 are in a corner.

   Without loss of generality, suppose that the 3×3 squares occupy the top left, top right, and bottom left corners. We will multiply by 4 for rotations after. We can ignore the 1×3 spaces in between the 3×3 squares because there is no way we can fit any part of a 3×3 square in there. We are left with a 4×4 grid in the bottom right corner of the board. There are 4 different spots the fourth 3×3 square can occupy, but one of those spots include touching a corner, so we can only place the fourth square in 3 ways. We have $3 \cdot 4 = 12$ cases here.

   Case 3: 2 squares occupy two adjacent corners.

   We will multiply by 4 for rotations later. Again, there is no way we can fit any part of a 3×3 square in the gap between 3×3 squares, so we are left with a grid that is 4 columns in width and 7 rows in length. Let the notation (a, b) refer to the grid cell in column a and row b, where the columns are numbered in increasing order from left to right and the rows are numbered in increasing order from down to up (like a coordinate system). We have that $(4, 1)$ and $(4, 7)$ are blocked off to prevent any more 3×3 squares from being in corners. If we try to use up some of the space in column 4, we have two arrangements for the center of the 3×3 squares: we can do $(2, 2)$ and $(3, 5)$ or we can do $(2, 5)$ and $(3, 2)$. However, if we don't want to use any of the space in column 4, we must place the squares in columns 1, 2, and 3. There are 3 ways to do so because it is the same as ordering 2 indistinguishable 3×3 squares and a 1×3 piece. So we have $4 \cdot 5 = 20$ cases here.

   Case 4: 2 squares occupy opposite corners.

   We will multiply by 2 for rotations at the end. We can think of the remaining portion of the board as two 4×4 grids that share a single cell in the center. Since the corners are blocked off, in a 4×4 grid, we have 3 ways to choose where the 3×3 square goes. We also have 3 ways to choose where the square goes in the other 3×3 square goes. However, one of the $3 \cdot 3$ ways involves both 3×3 squares overlapping at the center cell, so there are 8 ways to place the 3×3 squares safely. So we have $2 \cdot 8 = 16$ cases here.

Case 5: 1 square occupies a corner.

We will multiply by 4 for rotations at the end. Suppose the top left corner is the one that is occupied. Next, suppose we were to block off the remainder of the leftmost column. If we tried to block off the remainder of the second column as well, there would be no way to place the remaining three 3×3 squares. Therefore, we must place one of the 3×3 squares with center $(3, 2)$ or $(3, 3)$. In either case, the remaining two squares must have centers $(6, 3)$ and $(5, 6)$. If we didn't block off the leftmost column, we must have the first 3×3 square with center $(2, 3)$. Then, we have a 4×7 grid to work with and 2 of the corners are blocked off. Reusing our work from case 3, we have 5 ways to place the squares here. Therefore, we have $4 \cdot 7 = 28$ cases here.

Adding everything up, we have $1 + 12 + 20 + 16 + 28 = \boxed{077}$ total cases.

9. On each vertex of regular hexagon $ABCDEF$, where the vertices are distinct, a positive integer divisor of 2020 is written. Then, on each edge, the greatest common divisor of the two integers on the vertices containing the edge is written. Suppose the least common multiple of the six integers written on the edges is 2020. If N is the number of ways where this is possible, find the sum of the (not necessarily distinct) primes in the prime factorization of N.

Proposed by P_Groudon

(Answer: 363)

We note that $2020 = 2^2 \cdot 5 \cdot 101$. The divisors of 2020 will not be divisible by a prime that does not divide 2020. In addition, the greatest common factor and least common multiple functions merely change the exponents of the primes. Therefore, we will distribute each prime separately along the circle. Because 5 and 101 have the same exponent in the prime factorization of 2020, there will be the same number of ways to distribute them.

Let $v_p(2020)$ denote the number of times p shows up in the prime factorization of 2020. When we consider each prime separately, instead of writing divisors on each of the vertices of the hexagon, we write a number from 0 to $v_p(2020)$, inclusive.

When we take the greatest common factor of the numbers on two consecutive vertices, we write the smaller of the two exponents on the edge. When we take the least common multiple of the 6 numbers on the edges, we take the largest possible number among all the edges. In this case, 2020 must be the least common multiple. Therefore, for each prime p, we must have $v_p(2020)$ written on at least one of the edges. To achieve this, we must have two consecutive vertices with $v_p(2020)$ written on both.

Therefore, distributing a prime p is the same as: "Find the number of ways to write an integer from 0 to $v_p(2020)$, inclusive on each vertex of a regular hexagon such that there at least two consecutive vertices that both have $v_p(2020)$ written on it."

Let's consider the number of ways with $v_p(2020) = 1$. Let $C(k)$ be the number of arrangements that work if we have exactly k 1s in the circle.

By casework:

$C(2) = 6$, $C(3) = 18$, $C(4) = 15$, $C(5) = 6$, $C(6) = 1$. These add up to 46, so we have 46 ways to distribute the 5s and 101s.

Now we consider distributing the 2s by reusing some of our work from $v_p(2020) = 1$.

The number of ways to distribute the 2s is given by:

$$\sum_{k=2}^{6}(C(k)\cdot 2^{6-k}) = 313.$$

This is because for any valid arrangement with $v_p(2020) = 1$, we can change all the 1s to 2s. Then, for all the $6-k$ spaces with 0s, we can change them to either a 0 or a 1.

Therefore, $N = 313 \cdot 46^2$. We notice that 313 is prime, but $46 = 23 \cdot 2$.

Finally, $313 + 2 + 2 + 23 + 23 = \boxed{363}$.

10. It is given that the equation $x^3 + 2x^2 + 4x + 9 = 0$ has a unique real solution x such that

$$\lfloor 10^{11}x \rfloor = -211,785,097,233.$$

Find the sum of the digits of $\lfloor 10^{11}(x^4+4)\rfloor$. Note that $\lfloor r \rfloor$ denotes the greatest integer less than or equal to r for all real numbers r.

Proposed by Emathmaster

(Answer: 051)

Observe that $x^3 + 2x^2 + 4x + 8 = (x+2)(x^2+4)$. Subtracting 1 from both sides of $x^3 + 2x^2 + 4x + 9 = 0$, we get $(x+2)(x^2+4) = -1$. Multiplying both sides by $(x-2)$, we get $x^4 - 16 = -x + 2$. Adding 20 to both sides gives us $x^4 + 4 = -x + 22$. Therefore, the unique root x must satisfy $x^4 + 4 = -x + 22$.

We have that $\lfloor 10^{11}(x^4+4)\rfloor = \lfloor 10^{11}(-x+22)\rfloor$. Since $22 \cdot 10^{11}$ is an integer, we may take it out of the floors, which gives us $22 \cdot 10^{11} + \lfloor -10^{11}x \rfloor$. Observe that since $\lfloor 10^{11}x \rfloor$ is negative, we have that $\lfloor -10^{11}x \rfloor = -\lfloor 10^{11}x \rfloor - 1$.

Now, we compute $22 \cdot 10^{11} - \lfloor 10^{11}x \rfloor - 1$. We get that

$$22 \cdot 10^{11} - \lfloor 10^{11}x \rfloor - 1 = 2,200,000,000,000 + 211,785,097,233 - 1 = 2,411,785,097,232.$$

We get that the sum of the digits of this value is $\boxed{051}$.

11. Let ω be the incircle of $\triangle ABC$ and denote D and E as the tangency points of ω with sides $\overline{BC}$ and $\overline{AB}$, respectively. Line AD intersects ω at two distinct points, D and F. The circle passing through E that is tangent to line AD at F intersects line AB at two distinct points, E and G. Given that $AG < AE$, $AG = 24$, $EF = 20$, and $DE = 25$, length BE can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find the remainder when $m+n$ is divided by 1000.

Proposed by P_Groudon

(Answer: 417)

We will use the fact that for any circle, the angle θ formed by a tangent line and a chord of the circle that passed through the tangent line is half the angle of the bounded arc. This angle θ is equal to the angle measure of an inscribed angle subtending this same arc.

Denote the circumcircle of $\triangle EFG$ as ω_2.

First, we notice that $\angle AFG = \angle AEF$ because of the tangency of ω_2 with line AD. Therefore, $\triangle AGF \simeq \triangle AFE$. Using this, we can write the equation $\frac{GF}{FE} = \frac{AG}{AF}$. Because $AG = 24$ and $FE = 20$, this simplifies to $AF \cdot GF = 24 \cdot 20$.

Angle chasing further, we find that $\angle FEG = \angle FDE$ because of the tangency of ω with line AB. Because we found earlier that $\angle AFG = \angle AEF$ and $\angle AEF = \angle FEG$, $\overline{FG}$ is parallel to $\overline{DE}$. This means that $AGF \simeq AED$. We can write $\frac{AG}{GF} = \frac{AE}{ED}$. Because $AG = 24$ and $ED = 25$, this simplifies to $AE \cdot GF = 24 \cdot 25$.

Now, we divide the equation $AF \cdot GF = 24 \cdot 20$ that we found earlier by $AE \cdot GF = 24 \cdot 25$ to get $\frac{AF}{AE} = \frac{4}{5}$. Therefore, $AF = 4k$ and $AE = 5k$ for some positive real value k. Using Power of a Point with respect to ω_2, we have $AG \cdot AE = AF^2$. This equation becomes $24 \cdot 5k = 16k^2$. Solving for k, we get $k = \frac{15}{2}$. It then follows that $AF = 30$ and $AE = \frac{75}{2}$. Using Power of a Point with respect to ω, we have $AE^2 = AF \cdot (AF + FD)$. Plugging in our known values, the equation becomes $\frac{75^2}{2^2} = 30 \cdot (30 + FD)$. Solving for FD, we get $\frac{135}{8}$.

We can angle chase using the tangency of ω with $\overline{AB}$ and $\overline{BC}$ to find that $\angle EFD = \angle BED = \angle BDE$. Therefore, $\triangle BED$ is isosceles with base $\overline{ED}$. Call H the foot of the altitude from B to ED. Because H is the midpoint of $\overline{ED}$, we have $EH = \frac{25}{2}$. It then follows that $\cos(\angle BED) = \cos(\angle EFD) = \frac{25}{2 \cdot BE}$.

We know all the side lengths of $\triangle EFD$, which are $FD = \frac{135}{8}, EF = 20$, and $DE = 25$. Using these, we can find $\cos(\angle EFD)$. Since scaling the triangle preserves the cosine of the angle, we can scale the triangle by a factor of $\frac{8}{5}$ to make the side lengths all integers. After the scaling, $FD = 27, EF = 32$, and $DE = 40$. Using the reverse Law of Cosines, we have that $\cos(\angle EFD) = \frac{27^2+32^2-40^2}{2 \cdot 27 \cdot 32}$. However, $24^2 + 32^2 = 40^2$, which means that $-24^2 = 32^2 - 40^2$. As a result:

$$\cos(\angle EFD) = \frac{27^2 - 24^2}{2 \cdot 27 \cdot 32}$$
$$\cos(\angle EFD) = \frac{(27+24)(27-24)}{2 \cdot 27 \cdot 32}$$
$$\cos(\angle EFD) = \frac{51 \cdot 3}{2 \cdot 27 \cdot 32}$$
$$\cos(\angle EFD) = \frac{17}{192}$$

Because $\cos(\angle EFD) = \frac{25}{2 \cdot BE}$ or $BE = \frac{25}{2} \cdot \frac{1}{\cos(\angle EFD)}$, it follows that $BE = \frac{2400}{17}$. So, the remainder when $m + n$ is divided by 1000 is $400 + 17 = \boxed{417}$.

12. The value of

$$\frac{\sin^9\left(\frac{\pi}{18}\right) - 1}{5\sin\left(\frac{\pi}{18}\right) - 3\sin^2\left(\frac{\pi}{18}\right) - 15}$$

can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Find $m + n$.

Proposed by Emathmaster

(Answer: 137)

Let $x = \sin\left(\frac{\pi}{18}\right)$. By the triple sine formula, we have $\sin(3 \cdot \frac{\pi}{18}) = 3x - 4x^3 = \sin(\frac{\pi}{6}) = \frac{1}{2}$. Thus $8x^3 - 6x + 1 = 0$. Finally, computing

$$x^9 - 1 = (x^3)^3 - 1 = (\frac{6x-1}{8})^3 - 1 = \frac{216x^3 - 108x^2 + 18x - 1}{512} - 1$$

$$= \frac{27(8x^3) - 108x^2 + 18x - 1}{512} - 1 = \frac{27(6x-1) - 108x^2 + 18x - 1}{512} - 1 = \frac{180x - 108x^2 - 28}{512} - 1$$

$$= \frac{180x - 108x^2 - 540}{512} = \frac{9}{128}(5x - 3x^2 - 15).$$

So, $m + n = 9 + 128 = \boxed{137}$.

13. Let $\triangle ABC$ be an acute triangle with $BC = 2AC$. Let D be the midpoint of $\overline{BC}$ and E be the foot of the perpendicular from B to $\overline{AC}$. Lines BE and AD intersect at F such that $AF = 2CE$. The degree measure of angle C can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Proposed by AIME12345

(Answer: 547)

Let M be the midpoint of $\overline{AF}$. We have $ME = MA = CE$. We also have $DE = DC = DB = AC$, where the last step follows from the fact that we are given $BC = 2AC$. Using isosceles triangles $\triangle DCA$ and $\triangle AME$, we have that $\angle MEA = \angle EAM = \angle CAD = \angle CDA$. It then follows that quadrilateral $MCDE$ is cyclic.

Let $x = \angle CDA$, then $\angle AME = \angle DCA = 180^\circ - 2x$. Then $\angle CEM = 180^\circ - x$ and $\angle CME = 90^\circ - \frac{\angle CEM}{2} = \frac{x}{2}$. Then $\angle CDE = \angle CME = \frac{x}{2}$ so $\angle DCE = 90^\circ - \frac{\angle CDE}{2}$. Therefore $\angle DCA = 180^\circ - 2x = 90^\circ - \frac{x}{4}$ so $x = \frac{360}{7}$. Then $\angle DCA = 90^\circ - \frac{x}{4} = \frac{540}{7}$. Our answer is $m + n = 540 + 7 = \boxed{547}$.

14. Define the function

$$S(n) = \sum_{k=1}^{n} \left(k \left\lfloor \tfrac{n}{k} \right\rfloor\right)$$

for all positive integers n, where $\lfloor r \rfloor$ denotes the greatest integer less than or equal to r for all real numbers r. Find the sum of all positive integers n such that

$$S(2n) + S(n-1) - S(2n-1) - S(n) = 48.$$

Proposed by P_Groudon

(Answer: 080)

Define $\sigma(n)$ as the sum of the positive divisors of n. The idea is that we analyze the expression $k\lfloor \frac{n}{k} \rfloor$ combinatorically. The expression $\lfloor \frac{n}{k} \rfloor$ counts the number of integers less than n that are divisible by k. By multiplying this floor by k, we are adding k to our sum for every time k shows up as a divisor in the integers from 1 to n. Since we have k vary from 1 to n and no integer greater than n can be a divisor of the numbers 1 to n, this sums up every possible divisor of every possible integer from 1 to n. Therefore, we have that:

$$S(n) = \sum_{k=1}^{n} \left(k \left\lfloor \tfrac{n}{k} \right\rfloor\right) = \sum_{k=1}^{n} \sigma(k)$$

Therefore, $S(2n) + S(n-1) - S(2n-1) - S(n) = 48$ becomes $\sigma(2n) - \sigma(n) = 48$. Write n in the form $2^a b$, where a is a nonnegative integer and b is an odd integer. Because the function $\sigma(n)$ is multiplicative and 2^a and b are relatively prime, $\sigma(2n) = (1 + 2 + \cdots + 2^{a+1})\sigma(b)$ and $\sigma(n) = (1 + 2 + \cdots + 2^a)\sigma(b)$. Now our expression becomes $2^{a+1}\sigma(b) = 48$. Now we do casework on the value of a (while keeping in mind that b is odd).

Case 1: $a = 0$

This means that $\sigma(b) = 24$. First, we consider if b has three primes in its prime factorization. Since b is odd, we can't use 2 as a prime. The bare minimum for $\sigma(b)$ would be $(1+3)(1+5)(1+7) > 24$, so we can't have 3 primes in the prime factorization. If we consider two primes, the bare minimum would be $(1+3)(1+5) = 24$. This is the absolute bare minimum and increasing the exponent of any of the primes would increase the LHS. In this case, $(1+3)(1+5)$ corresponds to $b = 15$ or $n = 15$. If we consider 1 prime, we check if the exponent of the prime can be greater than 1. Obviously, $1 + 3 + 3^2 + \cdots + 3^k$ is relatively prime to 24 for all positive integers values of k and $1 + 5 + 5^2 > 24$ is too big. Therefore, we must have the exponent of the prime be 1 or $1 + p = 24$ for some prime p. This gives $p = 23$, which is valid and $n = 23$.

Case 2: $a = 1$

This means that $\sigma(b) = 12$. From our work above, since $(1+3)(1+5) = 24$ and because we will be increasing the value of a, which decreases the value of $\sigma(b)$, we can only have 1 prime in the prime factorization of b. Once again, $1 + 3 + 3^2 + \cdots + 3^k$ is relatively prime to 12 for all positive integers values of k and $1 + 5 + 5^2 > 12$ is too big. So we must have $1 + p = 12$, which gives $p = 11$. Therefore $b = 11$, $a = 1$, and $n = 22$.

Case 3: $a = 2$

This means that $\sigma(b) = 6$. Once again, $1 + 3 + 3^2 + \cdots + 3^k$ is relatively prime to 6 for all positive integers values of k and $1 + 5 + 5^2 > 6$ is too big. So we must have $1 + p = 6$. This gives $p = 5$, $b = 5$, $a = 2$, and $n = 20$.

Case 4: $a = 3$

This means that $\sigma(b) = 3$. We must have $1 + p = 3$. This gives $p = 2$, but this is impossible since b is odd. So there are no solutions in this case.

Alternatively, to locate values of odd integers x such that $\sigma(x) = 24$, $\sigma(x) = 12$, $\sigma(x) = 6$, and $\sigma(x) = 3$, one could use the fact that $\sigma(x) > x$ for all positive integers x. Therefore, one manually test the values of $\sigma(x)$ for odd integers x in the closed interval $[1, 23]$.

We have exhausted all cases, so the answer is $23 + 15 + 22 + 20 = \boxed{080}$.

15. In cyclic quadrilateral $ABCD$, $\overline{CD}$ is extended past C to intersect line AB at B', and $\overline{AD}$ is extended past D to intersect line BC at point D'. The circumcircles of $\triangle BB'C$ and $\triangle DD'C$ intersect at another point C', where $C' \neq C$. Given that $B'C' = 12$, $B'B = D'C' = 8$, and $D'D = 4$, length AC' can be expressed as $a\sqrt{b}$, where a and b are positive integers and b is not divisible by the square of any prime. Find $a + b$.

Proposed by Awesome_guy

(Answer: 040)

First, let us make the following claims:

Claim 1: B', C', and D' are collinear.

Proof: We know that since $ABCD$ is cyclic, $\angle ABC + \angle ADC = 180°$.

Note that since $CC'DD'$ is cyclic, $\angle D'C'C + \angle D'DC = 180°$. Also note that $\angle ADC + \angle D'DC = 180°$. Thus $\angle ADC = \angle D'C'C$.

Similarly, note that since $CC'BB'$ is cyclic, $\angle B'C'C + \angle B'BC = 180°$. Also note that $\angle ABC + \angle B'BC = 180°$. Thus $\angle ABC = \angle B'C'C$.

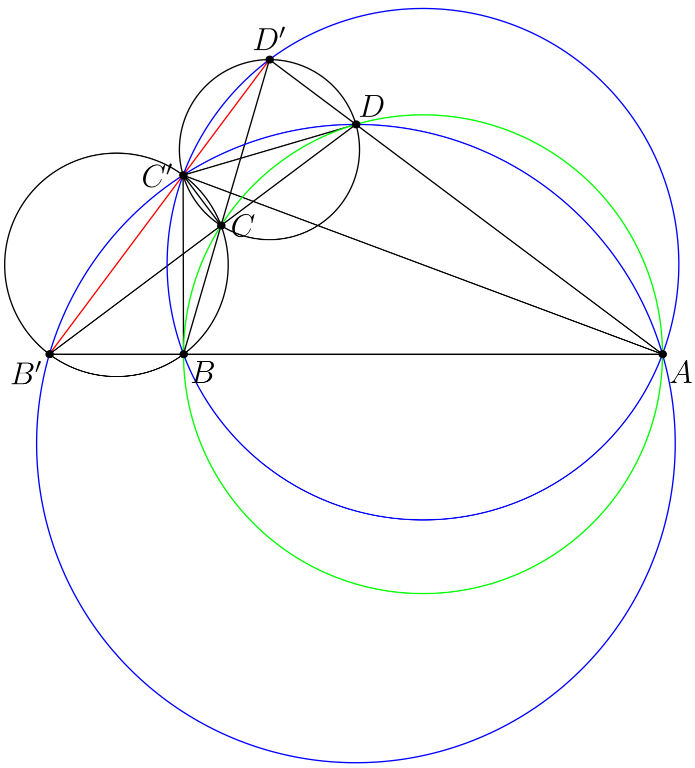
Substituting in, we have $\angle ABC + \angle ADC = \angle B'C'C + \angle D'C'C = 180°$, hence proved.

Claim 2: $AB'C'D$ is cyclic.

Proof: We know by definition $ABCD$ is cyclic. Thus $\angle BAD + \angle BCD = 180°$, or $\angle BAD = \angle D'CD$. We know that since $CC'DD'$ is cyclic and both $\angle D'CD$ and $\angle D'C'D$ subtend to $\widehat{\mathrm{DD}}'$, $\angle D'C'D = \angle BAD = \angle B'AD$. Thus $\angle B'C'D + \angle B'AD = 180°$, hence proved.

Claim 3: $ABC'D'$ is cyclic.

Proof: Similarly, we know by definition $ABCD$ is cyclic. Thus $\angle BAD + \angle BCD = 180°$, of $\angle BAD = \angle B'CB$. We know that since $CC'BB'$ is cyclic, $\angle B'CB$ and $\angle B'C'B$ subtend to $\widehat{\mathrm{BB}}'$, $\angle B'C'B = \angle BAD = \angle BAD'$. Thus $\angle BC'D' + \angle BAD' = 180°$, hence proved.



By Power of a Point with respect to the circumcircle of $AB'C'D$, we know $D'C' \cdot D'B' = D'D \cdot D'A$. Plugging in we have $8 \cdot 20 = 4 \cdot D'A$, and thus $D'A = 40$.

By Power of a Point with respect to the circumcircle of $ABC'D'$, we know $B'B \cdot B'A = B'C' \cdot B'D'$. Plugging in we have $8 \cdot B'A = 12 \cdot 20$, and thus $B'A = 30$.

Using Stewart's theorem with respect to $\triangle AB'D'$ and cevian $\overline{AC'}$, we know $AD'^2 \cdot B'C' + AB'^2 \cdot C'D' = AC'^2 \cdot B'D' + B'D' \cdot B'C' \cdot C'D'$. Plugging in we have

$$40^2 \cdot 12 + 30^2 \cdot 8 = AC'^2 \cdot 20 + 20 \cdot 12 \cdot 8$$

$$19200 + 7200 = AC'^2 \cdot 20 + 1920$$

$$AC'^2 = 1224$$

$$AC' = 6\sqrt{34}.$$

Thus $a + b = 6 + 34 = \boxed{040}$.