1st Olympiad Test Junior Mathematical Olympiad 2020

Language: **English**

Day: **1**

May 30th, 2020 to June 16th, 2020

Note: For any geometry problems whose statement begins with an asterisk ($*$), the first page of the solution must be a large, in-scale, clearly labeled diagram. Failure to meet this requirement will result in an automatic 1-point deduction.

J1. Let $\mathbb{Z}_{\geq 0}$ denote the set of non-negative integers. Find all functions $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ such that

$$f(m^2 + 2n) = (f(m))^2 + 2f(f(n))$$

holds true for all non-negative integers m and n.

J2. Find all positive integer ordered pairs (p, q), where p is a prime, that satisfy

$$\operatorname{lcm}(\phi(p), q) = p \cdot \phi(q).$$

For every positive integer k, $\phi(k)$ is the number of positive integers less than or equal to k that are relatively prime to k. Also, for every ordered pair of positive integers (a, b), $\operatorname{lcm}(a, b)$ is the least common multiple of a and b.

J3. ($*$) Given a segment AB and a circle Ω passing through points A and B, let C be an arbitrary point distinct from A and B on Ω. Let the external angle bisectors of $\angle BAC$ and $\angle ABC$ meet at a point I_C. Let D, E, and F be the feet of the perpendiculars from I_C onto lines BC, CA, and AB, respectively. Let H be the orthocenter of $\triangle DEF$. As C varies on Ω, let P be the fixed point on line HI_C.

Next, let ℓ be an arbitrary line which cuts segments BC and CA at points M and N, respectively. Let X, Y, and Z be the midpoints of segments AM, BN, and MN, respectively. Show that the foot of the perpendicular from P onto ℓ lies on the circumcircle of $\triangle XYZ$.

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.

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J4. Alice and Bob are playing a game with n distinct Lego bricks, labeled from 1 through n inclusive, where n is a positive integer. The game is played as such:

At the beginning, Alice randomly arranges the bricks into a line, with each possible ordering of bricks (with regards to the labels) having an equal chance of being chosen. Each move, Bob can take the leftmost brick in the line and place it in any other position (the position of a brick is uniquely defined by how many other bricks there are to the left of it), and his goal is to rearrange the bricks so that they lie in increasing order from left to right (so the leftmost brick would be 1 and the rightmost brick would be n).

For each n, compute the expected value of the minimum number of moves that Bob would require to achieve his goal.

J5. ($*$) In acute $\triangle ABC$ with circumcenter O and circumcircle Ω, line OB meets $\overline{AC}$ at E and meets Ω again at B'. Similarly, line OC meets $\overline{AB}$ at F and meets Ω again at C'. Let lines $B'F$ and $C'E$ meet at a point X. Line AX meets the altitude from B to $\overline{AC}$ at a point Y. The circles with diameters $\overline{BY}$ and $\overline{CY}$ meet Ω again at points B_1 and C_1, respectively. Prove that if line BC_1 meets line CB_1 at a point T, then points T, O, Y are collinear.

J6. For positive real numbers a, b, c with $abc = 1$, prove that

$$\frac{(a+b+c)^2}{a(b+c)(b^3+c^3)} + \frac{(a+b+c)^2}{b(a+c)(a^3+c^3)} + \frac{(a+b+c)^2}{c(a+b)(a^3+b^3)} \leq 3(a^3+b^3+c^3) - \frac{9}{4}.$$

Furthermore, determine the case when equality holds.

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.