OTSS TMC

Spring Mathematics Competitions

Spring Mathematics Competitions

1st Annual

TMC 10A

Thursday, April 30, 2020

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU TELL YOURSELF.
2. This is a twenty-five question multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. Mark your answer to each problem on the TMC 10 Answer Form with a keyboard. Check the keys for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded; however, this mock will be graded by people.
4. SCORING: You will receive **6** points for each correct answer, **1.5** points for each problem left unanswered, and **0** points for each incorrect answer.
5. No aids are permitted other than scratch paper, graph paper, rulers, compass, protractors, and erasers. No calculators, smartwatches, or computing devices are allowed. No problems on the test will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your proctor will not ask you to record certain information on the answer form.
8. When you give the signal, begin working on the problems. You will have 75 minutes to complete the test. You can discuss only with people that have taken the test during the period when make-ups are eligible.
9. When you finish the exam, don't sign your name in the space provided on the Answer Form.

The Committee on the Test Mathematics Competitions reserves the right to re-examine students before deciding whether to grant official status to their scores. The Committee also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

Students who score well on this TMC 10 will not be invited, but rather encouraged, to take the 2020 Olympiad Test Invitational Examination (OTIE) from Tuesday, May 26, 2020, to Thursday, June 4, 2020. More details about the OTIE and other information are not on the back page of this test booklet.

1. A right triangle has a leg of length 63 and a hypotenuse of length 65. What is its perimeter?

   (A) 140 **(B)** 142 **(C)** 144 **(D)** 146 **(E)** 148

2. Medium Z has 2 types of pets: Big Zs and Little Zs. Big Zs have 3 heads and 4 legs, and Little Zs have 2 heads and 2 legs. If there are total number of 29 heads and 34 legs among Medium Zs pets, then how many Little Zs does Medium Z have?

   (A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

3. Find the sum of the 20th, 202nd, and 2020th terms in this arithmetic sequence: $1, 5, 9, 13, \ldots$

   (A) 8955 **(B)** 8959 **(C)** 8963 **(D)** 8967 **(E)** 8971

4. What is the value of

$$\left(3+2\sqrt{2}\right)+\left(\frac{1}{3-2\sqrt{2}}\right)+\left(3-2\sqrt{2}\right)+\left(\frac{1}{3+2\sqrt{2}}\right)?$$

   (A) $4\sqrt{2}$ **(B)** 6 **(C)** $8\sqrt{2}$ **(D)** $6+4\sqrt{2}$ **(E)** 12

5. On a ten-question True/False test, Neel only knows the answer to three of the questions! As a result, he flips a fair coin to determine his answers for the rest of the questions. Given that a passing grade is anything above 60% in Neel's school and that he correctly answers all three questions where he knows the answer to, what is the probability that Neel passes the test?

   (A) $\frac{11}{64}$ **(B)** $\frac{29}{128}$ **(C)** $\frac{193}{512}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{99}{128}$

6. Given that a, b, and c are not necessarily distinct prime numbers, how many solutions are there to $a(a+b+c)=48$?

   (A) 2 **(B)** 4 **(C)** 5 **(D)** 7 **(E)** 8

7. Let ABC be an equilateral triangle. Next, let D be on the extension of $\overline{BC}$ past point B such that $\angle BAD = 30^\circ$, and let E be on the extension of $\overline{BC}$ past point C such that $\angle EAC = 30^\circ$. If $BC = 2$, what is the area of DAE?

   (A) $2\sqrt{2}$ **(B)** 3 **(C)** $2\sqrt{3}$ **(D)** $3\sqrt{3}$ **(E)** 6

8. What is the value of $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + 15 \cdot 16$?

   (A) 1320 **(B)** 1340 **(C)** 1360 **(D)** 1380 **(E)** 1400

9. Call an ordered pair of positive primes (a, b) *cool* if $a = b - 10$. Suppose that for some integer n, there exists a list of primes $P_1, P_2, \ldots, P_n$ such that (P_i, P_{i+1}) is *cool* for all $1 \leq i \leq n - 1$. What is the largest possible value of n?

   (A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 10

10. At Lexington High School, it is customary for people to not use adjacent stalls in a bathroom. Some (possibly empty) subset of five different kids want to use a row of four stalls at the same time. In how many ways can they do so?

    (A) 53 **(B)** 57 **(C)** 81 **(D)** 93 **(E)** 141

11. In how many ways can 720 be written as the product of two positive integers with a differing number of digits? (The order of the two integers does not matter.)

    (A) 3 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

12. To celebrate Bela's birthday, Jenn decides to make a cake in the shape of a right cylinder with a radius of 2 and a height of 10. Strangely, Jenn covers the entire outside (including the bottom) of the cake with frosting and cuts the cake such that each cut is parallel to the base of the cake, and each resulting slice is a cylinder. There is only sponge and no frosting on the inside of the cake. On each slice, Jenn wants the amount of frosting to be the same. If Jenn cuts the cake into 8 parts, what is the height of the slice that contains the top of the cake? (Assume the frosting has negligible thickness.)

    (A) $\frac{1}{4}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{2}{3}$ **(D)** 1 **(E)** $\frac{6}{5}$

13. Five students, Aidan, Andrew, Kevin, Eddie, and Andy, arm wrestle each other. Every two participants play against each other in exactly one match. How many possible sets of outcomes are there if Kevin wins every match he plays, Eddie loses every match he plays, and one of the five students wins exactly three matches?

    (A) 1 **(B)** 2 **(C)** 3 **(D)** 6 **(E)** 8

14. Every element in nonempty set S is a distinct nonnegative integer less than or equal to 16. The product of the elements is not divisible by 8 and there are at most 2 odd numbers in S. Let N be the number of possible sets that can be S. Find the sum of the digits of N.

    (A) 12 **(B)** 13 **(C)** 15 **(D)** 16 **(E)** 17

15. In an infinitely populated alternate world, Andrew has finally won a video game, and he is dying to tell everyone about it. On day 1, he calls 6 of his friends, and tells them about it. Each day afterwards, each person who was just informed calls 6 of their friends who have not been informed yet. Let n be the amount of informed people by the end of the 224th day, including Andrew. What is the remainder when n is divided by 11?

    (A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

16. Alice and Bob play a game. Alice goes first and they alternate between turns. In this game, an unfair coin is flipped. Alice wins if it is her turn and she flips heads; Bob wins if it is his turn and he flips tails. If the game is a fair game (i.e. both players have an equal chance of winning), what is the probability that the coin flips heads on a given flip?

    (A) $\frac{\sqrt{5}-1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{3-\sqrt{5}}{2}$ **(D)** $\frac{2}{5}$ **(E)** $\frac{1}{2}$

17. Let τ be a function such that for all positive integers n, $\tau(n)$ denotes the number of positive divisors n has. Given that there are two possible values of n such that $\tau(n+1) - \tau(n) \geq 14$, where $n < 200$, what is the sum of the digits of the smaller value of n?

    (A) 10 **(B)** 11 **(C)** 14 **(D)** 16 **(E)** 17

18. Let $s(n)$ denote the number of ways to partition n into two or more positive integers such that order matters (e.g. partitioning 10 into $3+7$ is different from $7+3$). The value of

$$s(1)+s(2)+\cdots+s(2020)$$

can be expressed as 2^a-b, where a and b are positive integers and $a+b$ is as small as possible. What is $10a+b$?

(A) 4041 **(B)** 22041 **(C)** 22221 **(D)** 22311 **(E)** 22441

19. Big Zhao and Little Zhao are playing a game where they take turns tiling a n by n plane with circular tiles of radius $\frac{n}{10}$ where $n \geq 20$. No tiles can overlap or go off the edge. A player wins in this game if the other player is unable to place a tile during their turn. If Big Zhao starts first, and both players play using optimal strategy, who will win?

(A) Little Zhao will always win. **(B)** Big Zhao will always win.
(C) Little Zhao will win if and only if $\left\lceil \frac{n}{\pi} \right\rceil$ is even.
(D) Big Zhao will win if and only if $\left\lceil \frac{n}{\pi} \right\rceil$ is even.
(E) Little Zhao will win if and only if $n \leq 100$.

20. There are M polynomials $P(x)$ such that, for all real values of x,

$$(x^3+x^2-4x-4)\cdot P(x) = (x-4)\cdot P(x^2),$$

and the leading coefficient of $P(x)$ is an integer with an absolute value of at most 5. Suppose the sum of all possible values of $P(3)$ is N. What is $M+N$?

(A) 1 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 264

21. Denote point C on circle ω with diameter $\overline{AB}$. The tangent lines to ω from A and C intersect at point D, with $BC=5$ and $CD=AD=3$. What is the length of $\overline{AB}$?

(A) 6 **(B)** $3\sqrt{5}$ **(C)** $5\sqrt{2}$ **(D)** $6\sqrt{2}$ **(E)** $9\sqrt{3}$

22. Given the polynomial x^3+6x^2+5x-7 with roots r_1, r_2, and r_3, what is the value of

$$\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)\cdot(r_1^2-3r_1+2)\cdot(r_2^2-3r_2+2)\cdot(r_3^2-3r_3+2)?$$

(A) 25 **(B)** 50 **(C)** 75 **(D)** 90 **(E)** 125

23. In convex quadrilateral $ABCD$, $\angle A = 90^\circ$, $\angle C = 60^\circ$, $\angle ABD = 25^\circ$, and $\angle BDC = 5^\circ$. Given that $AB = 4\sqrt{3}$, find the area of quadrilateral $ABCD$.

(A) 4 **(B)** $4\sqrt{3}$ **(C)** 8 **(D)** $8\sqrt{3}$ **(E)** $16\sqrt{3}$

24. $\triangle ABC$ has $AB = 49$, $AC = 35$, and $BC = 56$. Denote D as the foot of the altitude from A to $\overline{BC}$, E as the foot of the altitude from B to $\overline{AC}$, and F as the foot of the altitude from C to $\overline{AB}$. Let G be the intersection of $\overline{EF}$ and $\overline{AD}$. Let $EG : GF$ be in the form $\frac{m}{n}$ where m and n are relatively prime integers. What is $m + n$?

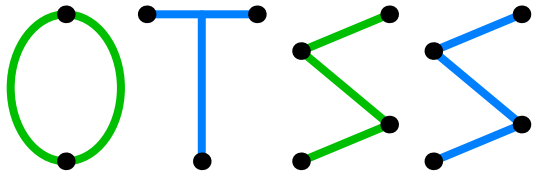
(A) 100 **(B)** 101 **(C)** 102 **(D)** 103 **(E)** 104

25. Define $s(k)$ as the period of the decimal expansion of $\frac{1}{k}$. Let S be the set of integers that are greater than 1 and can be written in the form $3^a \cdot 7^b$, where a and b are nonnegative integers. What is the value of $\frac{1}{s(k)}$ summed over all $k \in S$?

(A) $\frac{19}{6}$ **(B)** $\frac{229}{72}$ **(C)** $\frac{27}{8}$ **(D)** $\frac{11}{3}$ **(E)** $\frac{271}{72}$

2020 TMC 10

DO NOT OPEN UNTIL THURSDAY, April 30, 2020



Olympiad Test Spring Series

Questions and comments about problems and solutions for this exam should be sent by PM to:

Emathmaster and **kevinmathz**.

The 2020 OTIE will be held from Tuesday, May 26, 2020, to Thursday, June 4, 2020. It is a 15-question, 3-hour, integer-answer exam. You will not be invited, but rather encouraged, to participate based on your score on this competition. The top scoring students from both the TMC and the OTIE will not be invited, but rather encouraged, to take the 2020 Olympiad Test Junior Math Olympiad (OTJMO) from Saturday, May 30, 2020, to Tuesday, June 16, 2020. *A complete listing of our previous publications may be found at our web site:*

`https://online-test-seasonal-series.github.io`

****Administration On An Earlier Date Will Literally Be Impossible****

1. All the information needed to administer this exam is contained in the non-existent TMC 10/12 Teacher's Manual.
2. YOU must not verify on the non-existent TMC 10/12 COMPETITION CERTIFICATION FORM that you followed all rules associated with the administration of the exam.
3. All TMC 10 Answer Sheets must be returned to OTSS a week after the competition. Ship with inappropriate postage without using a tracking method. FedEx or AoPS is strongly recommended.
4. The publication, reproduction, or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, Discord, Facebook, Hangouts or other digital media of any type during this period is a violation of the competition rules.

The **2020 Olympiad Spring Tests**
was made possible by the contributions of the following people:

AIME12345, andyxpandy99, Awesome_guy, AwesomeYRY, azduncan, DeToasty3, Emathmaster, Eyed, I-_-I, Ish_Sahh, ivyzheng, jeteagle, kevinmathz, kvs, NJOY, PCChess, P_Groudon, Radio2, realquarterb, Stormersyle, VulcanForge, and zhao_andrew

Finally, we thank you for taking this mock. We hope you enjoyed it!