2020 TMC 10A Problems and Solutions Document

Olympiad Test Spring Series

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1. A right triangle has a leg of length 63 and a hypotenuse of length 65. What is its perimeter?

   (A) 140 **(B)** 142 **(C)** 144 **(D)** 146 **(E)** 148

   Proposed by Emathmaster

   Answer (C): Applying Pythagorean's Theorem on the triangle gives the length of the third side as
   $$\sqrt{65^2 - 63^2} = \sqrt{(65+63)(65-63)} = \sqrt{128(2)} = \sqrt{(2^7)(2)} = 16.$$
   Therefore, the perimeter is $65 + 63 + 16 = \boxed{\textbf{(C) } 144}$.

2. Medium Z has 2 types of pets: Big Zs and Little Zs. Big Zs have 3 heads and 4 legs, and Little Zs have 2 heads and 2 legs. If there are total number of 29 heads and 34 legs among Medium Zs pets, then how many Little Zs does Medium Z have?

   (A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

   Proposed by jeteagle

   Answer (E): Let x and y be the number of Big Zs and Little Zs that Medium Z has, respectively. We are given the two following equations: $3x + 2y = 29$ and $4x + 2y = 34$. Subtracting the first from the second, we get $x = 5$. Plugging this into our first equation gives $15 + 2y = 29$, so $y = \boxed{\textbf{(E) } 7}$.

3. Find the sum of the 20th, 202nd, and 2020th terms in this arithmetic sequence: $1, 5, 9, 13, \ldots$

   (A) 8955 **(B)** 8959 **(C)** 8963 **(D)** 8967 **(E)** 8971

   Proposed by jeteagle

   Answer (B): Each term in this sequence can be expressed in the form $4k-3$, where k denotes which term it is. Plugging this in for 20, 202, and 2020 gives us $77+805+8077 = \boxed{\textbf{(B) } 8959}$.

4. What is the value of

$$\left(3+2\sqrt{2}\right)+\left(\frac{1}{3-2\sqrt{2}}\right)+\left(3-2\sqrt{2}\right)+\left(\frac{1}{3+2\sqrt{2}}\right)?$$

(A) $4\sqrt{2}$ **(B)** 6 **(C)** $8\sqrt{2}$ **(D)** $6+4\sqrt{2}$ **(E)** 12

Proposed by Emathmaster

Answer (E): To remove the radicals from the denominator, we multiply the denominator and numerator by the conjugate:

$$\left(\frac{1}{3-2\sqrt{2}}\right)\cdot\left(\frac{3+2\sqrt{2}}{3+2\sqrt{2}}\right)=3+2\sqrt{2}$$

Similarly,

$$\left(\frac{1}{3+2\sqrt{2}}\right)=3-2\sqrt{2}$$

Adding everything up gives us $\boxed{\textbf{(E) } 12}$.

5. On a ten-question True/False test, Neel only knows the answer to three of the questions! As a result, he flips a fair coin to determine his answers for the rest of the questions. Given that a passing grade is anything above 60% in Neel's school and that he correctly answers all three questions where he knows the answer to, what is the probability that Neel passes the test?

(A) $\frac{11}{64}$ **(B)** $\frac{29}{128}$ **(C)** $\frac{193}{512}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{99}{128}$

Proposed by kevinmathz

Answer (D): Neel needs at least 4 of the last 7 because a passing grade is above 60%. Thus, by symmetry, getting 3 and 4 are symmetric, and so are 2 and 5, 1 and 6, and 0 and 7, so the probability Neel passes the test is $\boxed{\textbf{(D) } \frac{1}{2}}$.

6. Given that a, b, and c are not necessarily distinct prime numbers, how many solutions are there to $a(a+b+c)=48$?

(A) 2 **(B)** 4 **(C)** 5 **(D)** 7 **(E)** 8

Proposed by PCChess

Answer (D): We see that $48=2^4\cdot 3$, so that means that $a=2$ or $a=3$ because a has to be a prime.

If $a=2$, then $a+b+c=24\to 2+b+c=24\to b+c=22$. The following pairs work: $(3,19),(5,17),(11,11),(17,5),(19,3)$. This contributes 5 pairs to our total.

If $a=3$, then $a+b+c=16\to 3+b+c=16\to b+c=13$. The following pairs work: $(2,11),(11,2)$. This contributes 2 more pairs to our total.

Thus, our answer is $5+2=\boxed{\textbf{(D) } 7}$.

7. Let ABC be an equilateral triangle. Next, let D be on the extension of $\overline{BC}$ past point B such that $\angle BAD = 30°$, and let E be on the extension of $\overline{BC}$ past point C such that $\angle EAC = 30°$. If $BC = 2$, what is the area of DAE?

   (A) $2\sqrt{2}$ **(B)** 3 **(C)** $2\sqrt{3}$ **(D)** $3\sqrt{3}$ **(E)** 6

   Proposed by Ish_Sahh

   Answer (D): We see that the base of $\triangle DAE$ has length 6 and the height has length $\sqrt{3}$. Thus, the area is $\frac{1}{2} \cdot 6 \cdot \sqrt{3} = \boxed{\textbf{(D) } 3\sqrt{3}}$.

8. What is the value of $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + 15 \cdot 16$?

   (A) 1320 **(B)** 1340 **(C)** 1360 **(D)** 1380 **(E)** 1400

   Proposed by Awesome_guy

   Answer (C): We know that

$$\sum_{n=0}^{15} n \cdot (n+1) = \sum_{n=0}^{15} n^2 + n$$

   Using the formulas

$$\sum_{n=0}^{k} n^2 = \frac{k(k+1)(2k+1)}{6} \text{ and } \sum_{n=0}^{k} n = \frac{k(k+1)}{2}$$

   we can plug in $k = 15$:
   $= \sum_{n=0}^{15} n^2 + n = \frac{15(16)(31)}{6} + \frac{15(16)}{2} = 1240 + 120 = \boxed{\textbf{(C) } 1360}$.

9. Call an ordered pair of positive primes (a, b) *cool* if $a = b - 10$. Suppose that for some integer n, there exists a list of primes $P_1, P_2, \ldots, P_n$ such that (P_i, P_{i+1}) is *cool* for all $1 \le i \le n-1$. What is the largest possible value of n?

   (A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 10

   Proposed by PCChess

   Answer (B): If $n \ge 3$, it is guaranteed that at least one number is divisible by 3. This means that n can be maximized by making $P_1 = 3$. Checking, the list of numbers $3, 13, 23$ works, so the answer is $\boxed{\textbf{(B) } 3}$.

10. At Lexington High School, it is customary for people to not use adjacent stalls in a bathroom. Some (possibly empty) subset of five different kids want to use a row of four stalls at the same time. In how many ways can they do so?

(A) 53 **(B)** 57 **(C)** 81 **(D)** 93 **(E)** 141

Proposed by Emathmaster

Answer (C): We do casework based on how many stalls are occupied.

Case 1: No stalls occupied

There is 1 case here.

Case 2: 1 stall occupied

We may choose any of the 5 students and any of the 4 stalls. So we have $5 \cdot 4 = 20$ cases here.

Case 3: 2 stalls occupied

In order for no two adjacent stalls to be occupied, we can either have stalls 1 and 3 occupied, stalls 2 and 4 occupied, or stalls 1 and 4 occupied. This gives 3 different acceptable configurations. We may choose any of the 5 students to go in the left stall and then any 4 of the remaining students in the other stall. This gives $3 \cdot 5 \cdot 4 = 60$ cases here.

Case 4: 3 stalls occupied

This would require some two adjacent stalls, so we cannot have 3 stalls occupied.

In total, this gives us $1 + 20 + 60 = \boxed{\textbf{(C) } 81}$ ways.

11. In how many ways can 720 be written as the product of two positive integers with a differing number of digits? (The order of the two integers does not matter.)

(A) 3 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Proposed by Emathmaster

Answer (D): We realize that one of the numbers must have 1 digit and the other must be either 2 or 3 digits. Thus listing them out gives us

$$1 \cdot 720$$

$$2 \cdot 360$$

$$3 \cdot 240$$

$$4 \cdot 180$$

$$5 \cdot 144$$

$$6 \cdot 120$$

$$8 \cdot 90$$

$$9 \cdot 80$$

Therefore, there are a total of $\boxed{\textbf{(D) } 8}$ ways.

12. To celebrate Bela's birthday, Jenn decides to make a cake in the shape of a right cylinder with a radius of 2 and a height of 10. Strangely, Jenn covers the entire outside (including the bottom) of the cake with frosting and cuts the cake such that each cut is parallel to the base of the cake, and each resulting slice is a cylinder. There is only sponge and no frosting on the inside of the cake. On each slice, Jenn wants the amount of frosting to be the same. If Jenn cuts the cake into 8 parts, what is the height of the slice that contains the top of the cake? (Assume the frosting has negligible thickness.)

    (A) $\frac{1}{4}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{2}{3}$ **(D)** 1 **(E)** $\frac{6}{5}$

    Proposed by PCChess

    Answer (B): The surface area is $2 \cdot 2^2\pi + 2\pi 2 \cdot 10 = 48\pi$. This means that each slice has 6π of frosting. Solving $6\pi = 2^2\pi + 2\pi 2h$, we get that $h = \boxed{\textbf{(B)}\ \frac{1}{2}}$.

13. Five students, Aidan, Andrew, Kevin, Eddie, and Andy, arm wrestle each other. Every two participants play against each other in exactly one match. How many possible sets of outcomes are there if Kevin wins every match he plays, Eddie loses every match he plays, and one of the five students wins exactly three matches?

    (A) 1 **(B)** 2 **(C)** 3 **(D)** 6 **(E)** 8

    Proposed by kevinmathz

    Answer (D): Take away Kevin and Eddie - someone has to be first and win 2 games. Thus someone must win 1 and 0 games since there are 3 games without Kevin and Eddie. There are thus $3 \cdot 2 \cdot 1 = \boxed{\textbf{(D)}\ 6}$ ways to make the outcomes.

14. Every element in nonempty set S is a distinct nonnegative integer less than or equal to 16. The product of the elements is not divisible by 8 and there are at most 2 odd numbers in S. Let N be the number of possible sets that can be S. Find the sum of the digits of N.

    (A) 12 **(B)** 13 **(C)** 15 **(D)** 16 **(E)** 17

    Proposed by kevinmathz

    Answer (A): Since S is not divisible by 8, the multiples of 8 are out of consideration. Now we check evens and multiples of 4. At the end, we multiply by $\binom{8}{0} + \binom{8}{1} + \binom{8}{2} = 37$.

    Evens not divisible by 4: We have $2, 6, 10, 14$. Multiples of 4 but not 8: We have $4, 12$.

    Our number can be odd, or even but not divisible by 8. Thus, we add our number of ways to choose this, which is thus $\binom{4}{2} + \binom{4}{1} + \binom{2}{1} + 1 = 13$. Our answer is thus $37 \cdot 13 - 1 = 480$. We subtract by 1 because we cannot include the empty subset. The sum digits of sum of digits is $\boxed{\textbf{(A)}\ 12}$.

15. In an infinitely populated alternate world, Andrew has finally won a video game, and he is dying to tell everyone about it. On day 1, he calls 6 of his friends, and tells them about it. Each day afterwards, each person who was just informed calls 6 of their friends who have not been informed yet. Let n be the amount of informed people by the end of the 224th day, including Andrew. What is the remainder when n is divided by 11?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

Proposed by azduncan

Answer (A): We notice that at the end of day $n > 1$, 6^n new people become informed. Therefore, at the end of day 224, we have $1 + 6 + 6^2 + ... + 6^{224}$ people informed.

By the sum of a geometric series, the sum becomes $\frac{6^{225}-1}{5}$. We want to find this modulo 11.

A quick check shows that under modulo 11, the sequence $6, 6^2, 6^3, 6^4, ...$ is periodic every 10 terms. Therefore, $6^{225} \equiv 6^5 \pmod{11}$. We can also quickly check that $6^5 \equiv -1 \pmod{11}$. Therefore, $\frac{6^{225}-1}{5} \equiv \frac{-2}{5} \pmod{11}$. Adding $11 \cdot 2$ to the numerator of $\frac{-2}{5}$, we get $\frac{-2}{5} \equiv \frac{20}{5} \equiv 4 \pmod{11}$. Therefore, the answer is $\boxed{\textbf{(A) } 4}$.

16. Alice and Bob play a game. Alice goes first and they alternate between turns. In this game, an unfair coin is flipped. Alice wins if it is her turn and she flips heads; Bob wins if it is his turn and he flips tails. If the game is a fair game (i.e. both players have an equal chance of winning), what is the probability that the coin flips heads on a given flip?

(A) $\frac{\sqrt{5}-1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{3-\sqrt{5}}{2}$ **(D)** $\frac{2}{5}$ **(E)** $\frac{1}{2}$

Proposed by kevinmathz

Answer (C): Let p be the probability that the coin flips heads. The probability Alice wins can be written as $p + pr + pr^2 + pr^3 + \cdots$, where $r = \text{ratio} = p(1-p)$ because Tail-Heads has to be rolled to cycle. With the formula, we see that we have the total probability as $\frac{p}{1-p(1-p)} = \frac{1}{2}$, so thus, $p^2 - 3p + 1 = 0$, so using the quadratic formula gets $p = \frac{3\pm\sqrt{5}}{2}$. Since our answer is less than 1, it is $\boxed{\textbf{(C) } \dfrac{3-\sqrt{5}}{2}}$.

17. Let τ be a function such that for all positive integers n, $\tau(n)$ denotes the number of positive divisors n has. Given that there are two possible values of n such that $\tau(n+1) - \tau(n) \geq 14$, where $n < 200$, what is the sum of the digits of the smaller value of n?

(A) 10 **(B)** 11 **(C)** 14 **(D)** 16 **(E)** 17

Proposed by Emathmaster

Answer (C): Clearly, $n = 1$ and $n = 2$ does not satisfy the inequality. Therefore, we will assume that $\tau(n) \geq 2$. This means that $\tau(n+1) \geq 16$. We will first search for integers with $\tau(n+1) = 16$. Note that $n+1$ is even if we elect for $\tau(n+1) = 16$ because that implies $\tau(n) = 2$, which means n is prime, and since 2 doesn't work, it must be an odd prime.

Now, we search for integers whose prime factorizations are in the form p^{15}, $p^7 \cdot q$, $p^3 \cdot q^3$, $p^3 \cdot q \cdot r$, and $p \cdot q \cdot r \cdot s$. We'll focus more on the last 2, since they will most likely help us minimize n. We realize that $2 \cdot 3 \cdot 5 \cdot 7 = 210$ is too big. We also realize that $2^3 \cdot 3 \cdot 5 - 1 = 119$ is also not prime. However, $2^3 \cdot 3 \cdot 7 - 1 = 167$ is prime.

We claim that the smallest possible value of n is 167.

Now, we will look at the other prime factorizations that give us $\tau(n+1) = 16$. Clearly, 2^{15} is much bigger than 200. $2^7 \cdot 3 = 384$ is also too big. $2^3 \cdot 3^3 = 216$ is also too big. If we try $n + 1 = p^3 \cdot q \cdot r$ with $q < r$, first we will assume $p = 2$ then assume $q = 2$. If $p = 2$ and $q > 3$, then $2^3 \cdot 5 \cdot 7 = 280 > 200$, so $p = 2$ implies $q = 3$. Since we already tested $2^3 \cdot 3 \cdot 5$ and $2^3 \cdot 3 \cdot 7$, we test $q = 11$. However, $2^3 \cdot 3 \cdot 11 = 8 \cdot 33 > 200$ is too big. If we assume $q = 2$, then $n + 1$ has to be at least $3^3 \cdot 2 \cdot 5 = 270 > 200$, which is a contradiction. Therefore, there are no other solutions with $\tau(n + 1) = 16$ with $n < 200$.

If we have $\tau(n+1) = 17$, then $n+1 = p^{16}$ at the minimum. However, 2^{16} is much bigger than 200, so we suspect for a solution in $\tau(n + 1) = 18$.

The minimum possible value of $n+1$ with $\tau(n+1) = 18$ must be $n+1 = p^2 \cdot q^2 \cdot r = 2^2 \cdot 3^2 \cdot 5 = 180$. Because $n = 179$, which is prime, $n = 179$ must be another solution. Since we are given that there are two solutions, $n = 167$ and $n = 179$ are the two solutions in question. The smaller is 167, which has a digit sum of $\boxed{\textbf{(C) } 14}$.

18. Let $s(n)$ denote the number of ways to partition n into two or more positive integers such that order matters (e.g. partitioning 10 into $3 + 7$ is different from $7 + 3$). The value of

$$s(1) + s(2) + \cdots + s(2020)$$

can be expressed as $2^a - b$, where a and b are positive integers and $a+b$ is as small as possible. What is $10a + b$?

(A) 4041 **(B)** 22041 **(C)** 22221 **(D)** 22311 **(E)** 22441

Proposed by PCChess

Answer (C): Consider n stars arranged in a line. Place a divider directly left of the leftmost star and another divider directly right of the rightmost star. For each of the $n - 1$ spaces between two consecutive stars, we may choose to insert a divider. The number of stars between any two consecutive dividers determine the numbers for our partition from left to right order.

For example, if we have 1 divider, 4 stars, 1 divider, 3 stars, 1 divider, and 2 stars in that order, we are representing the partition $4 + 3 + 2$. We have 2^{n-1} ways to place the dividers in between any two stars, since we can have a divider or not. However, if we opt for no extra dividers, then we get one number in the partition, which doesn't work. All other partitions give two or more numbers. Therefore, $s(n) = 2^{n-1} - 1$ and:

$$\sum_{i=1}^{2020} s(i) = 1 + 2 + 4 + 8 + \cdots + 2^{2019} - 2020 = 2^{2020} - 2021.$$

Therefore, our answer is $10 \cdot 2020 + 2021 = \boxed{\textbf{(C) } 22221}$.

19. Big Zhao and Little Zhao are playing a game where they take turns tiling a n by n plane with circular tiles of radius $\frac{n}{10}$ where $n \geq 20$. No tiles can overlap or go off the edge. A player wins in this game if the other player is unable to place a tile during their turn. If Big Zhao starts first, and both players play using optimal strategy, who will win?

(A) Little Zhao will always win. **(B)** Big Zhao will always win.
(C) Little Zhao will win if and only if $\lceil \frac{n}{\pi} \rceil$ is even.
(D) Big Zhao will win if and only if $\lceil \frac{n}{\pi} \rceil$ is even.
(E) Little Zhao will win if and only if $n \leq 100$.

Proposed by jeteagle

Answer (B): We will prove Big Zhao will always win. First, let Big Zhao tile the center of this n by n plane. Now, every time Little Zhao places his tile, Big Zhao can tile his tile directly opposite of it from the center of the plane. If Little Zhao is able to tile his tile at some place, then Big Zhao will also because the tiling is symmetric across the center. Therefore, Big Zhao will be able to mirror Little Zhao's placements, and he will never run out of tiling places unless Little Zhao runs out first. This means $\boxed{\textbf{(B)}\text{ Big Zhao will always win.}}$

20. There are M polynomials $P(x)$ such that, for all real values of x,

$$(x^3 + x^2 - 4x - 4) \cdot P(x) = (x - 4) \cdot P(x^2),$$

and the leading coefficient of $P(x)$ is an integer with an absolute value of at most 5. Suppose the sum of all possible values of $P(3)$ is N. What is $M + N$?

(A) 1 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 264

Proposed by kevinmathz and Awesome_guy

Answer (D): We can factor $x^3 + x^2 - 4x - 4$ as follows:

$$x^2(x+1) - 4(x+1)$$

$$(x^2 - 4)(x+1)$$

$$(x+2)(x-2)(x+1)$$

Therefore, $(x+2)(x-2)(x+1) \cdot P(x) = (x-4) \cdot P(x^2)$.
Plugging in $x = \pm 2$, we get $0 = P(4)$. Therefore, $P(x) = Q(x)(x-4)$ for some polynomial $Q(x)$. Substituting this back into our equation, we get:

$$(x+2)(x-2)(x+1)(x-4) \cdot Q(x) = (x-4)(x^2-4) \cdot Q(x^2)$$

Cancelling the common factors on both sides, we get:

$$(x+1) \cdot Q(x) = Q(x^2)$$

Plugging in $x = -1$ tells us that $0 = Q(1)$. Therefore, $Q(x) = (x-1)R(x)$ for some polynomial $R(x)$. Substituting this back and cancelling common factors, we get:

$$R(x) = R(x^2)$$

Since this relation must hold for all real x and because the degree of $R(x^2)$ is double the degree of $R(x)$, we must have $R(x)$ be a constant polynomial. Let $R(x) = a$ for some real number a. Retracing our steps:

$$P(x) = a(x-4)(x-1)$$

The leading coefficient in $P(x)$ is a, so a can be any integer from -5 to 5, inclusive. This counts the zero polynomial as well. Therefore, $M = 11$.

We also have that $P(3) = -2a$, but the sum of the integers from -5 to 5, inclusive is 0. Therefore, $N = 0$, so $M + N = \boxed{\textbf{(D) } 11}$.

21. Denote point C on circle ω with diameter $\overline{AB}$. The tangent lines to ω from A and C intersect at point D, with $BC = 5$ and $CD = AD = 3$. What is the length of $\overline{AB}$?

(A) 6 **(B)** $3\sqrt{5}$ **(C)** $5\sqrt{2}$ **(D)** $6\sqrt{2}$ **(E)** $9\sqrt{3}$

Proposed by Awesome_guy

Answer (B): Denote r as the radius of the circle. We seek the value of $2r$. Define O as the center of the circle. Denote E as the foot of the altitude from O to BC. E is the midpoint of BC.

Clearly, DO bisects $\angle ADC$. Let $\angle ADO = \angle ODC = \theta$. Because $\angle DAO = \angle DCO = 90$, we have that $\angle AOD = \angle DOC = 90 - \theta$. Because $\angle AOC + \angle COB = 180$ and OE bisects $\angle COB$, we have that $\angle COE = \theta$. Because $\angle DAO = \angle OEC = 90$, it follows that $\triangle DAO \simeq \triangle OEC$. We then proceed to length chase.

We know that $CE = \frac{5}{2}$ and $CO = r$. We also know that $DA = 3$ and $AO = r$. Through Pythagorean Theorem on $\triangle DAO$, we have that $DO = \sqrt{r^2+9}$. Using the similar triangles, $\frac{DO}{OA} = \frac{OC}{CE}$. Plugging in the known lengths and clearing denominators, we arrive at $2r^2 = 5\sqrt{r^2+9}$. After squaring both sides and moving all the terms to one side, we get $4r^4 - 25r^2 - 225 = 0$. Solving the quadratic in terms of r^2, we find that $r^2 = \frac{45}{4}$ (disregarding the negative root). It then follows that $r = \frac{3\sqrt{5}}{2}$ and our answer is $\boxed{\textbf{(B) } 3\sqrt{5}}$.

22. Given the polynomial $x^3 + 6x^2 + 5x - 7$ with roots r_1, r_2, and r_3, what is the value of

$$\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right) \cdot (r_1^2 - 3r_1 + 2) \cdot (r_2^2 - 3r_2 + 2) \cdot (r_3^2 - 3r_3 + 2)?$$

(A) 25 **(B)** 50 **(C)** 75 **(D)** 90 **(E)** 125

Proposed by Ish_Sahh and Emathmaster

Answer (E): Let the polynomial be known as $P(x)$ with $P(x) = x^3 + 6x^2 + 5x - 7 = (x-r_1)(x-r_2)(x-r_3)$. To compute $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$, we can rewrite it as $\frac{r_1r_2 + r_1r_3 + r_2r_3}{r_1r_2r_3}$. Using Vieta's formula, we can get this as $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{5}{7}$. Next we can take the factor $(r_1^2 - 3r_1 + 2)$ and rewrite it as $(r_1 - 2)(r_1 - 1)$ using factoring. Similarly, $(r_2^2 - 3r_2 + 2) = (r_2 - 2)(r_2 - 1)$ and

$(r_3^2-3r_3+2) = (r_3-2)(r_3-1)$. This makes the original expression $(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3})(r_1^2-3r_1+2)(r_2^2-3r_2+2)(r_3^2-3r_3+2)? = \frac{5}{7}(r_1-2)(r_1-1)(r_2-2)(r_2-1)(r_3-2)(r_3-1)$. Next, if (r_1-2), (r_2-2), and (r_3-2) are grouped then they would give $(r_1-2)(r_2-2)(r_3-2) = -(2-r_1)(2-r_2)(2-r_3) = -P(2)$. Similarly, $(r_1-1)(r_2-1)(r_3-1) = -(1-r_1)(1-r_2)(1-r_3) = -P(1)$. This makes the expression to compute be $\frac{5}{7}(-P(1))(-P(2)) = \frac{5}{7}(P(1))(P(2)) = \frac{5}{7} \times 5 \times 35 = \boxed{\textbf{(E) } 125}$.

23. In convex quadrilateral $ABCD$, $\angle A = 90°$, $\angle C = 60°$, $\angle ABD = 25°$, and $\angle BDC = 5°$. Given that $AB = 4\sqrt{3}$, find the area of quadrilateral $ABCD$.

$\textbf{(A) } 4 \qquad \textbf{(B) } 4\sqrt{3} \qquad \textbf{(C) } 8 \qquad \textbf{(D) } 8\sqrt{3} \qquad \textbf{(E) } 16\sqrt{3}$

Proposed by DeToasty3

Answer (D): We claim that if we reflect point C across the perpendicular bisector of line segment $\overline{BD}$ to get point C', then we get a right triangle ABC', where point D is on side AC'. We see that this happens because $\angle ABC' = \angle BDC' + \angle ABD = 5° + 25° = 30°$, $\angle BC'D = \angle BC'A = 60°$, and $\angle BAC' = 90°$. We also know that $\angle ADC' = \angle BDA + \angle BDC' = (180° - 90° - 25°) + (180° - 60° - 5°) = 65° + 115° = 180°$, so point D is on side AC'. By extension, we now know that right triangle ABC' is a $30-60-90$ right triangle, where $\angle A = 90°$, $\angle B = 30°$, and $\angle C' = 60°$.

We know that right triangle ABC' has the same area as quadrilateral $ABCD$ because triangles BCD and $BC'D$ have the same areas (this reflection preserves areas), and triangle ABD is unchanged. Since we are given that $AB = 4\sqrt{3}$, it follows that the other leg, AC', has length 4. We have that the area of right triangle ABC', and thereby the area of quadrilateral $ABCD$, is $\frac{1}{2} \cdot 4 \cdot 4\sqrt{3} = \boxed{\textbf{(D) } 8\sqrt{3}}$.

24. $\triangle ABC$ has $AB = 49$, $AC = 35$, and $BC = 56$. Denote D as the foot of the altitude from A to $\overline{BC}$, E as the foot of the altitude from B to $\overline{AC}$, and F as the foot of the altitude from C to $\overline{AB}$. Let G be the intersection of $\overline{EF}$ and $\overline{AD}$. Let $EG : GF$ be in the form $\frac{m}{n}$ where m and n are relatively prime integers. What is $m+n$?

$\textbf{(A) } 100 \qquad \textbf{(B) } 101 \qquad \textbf{(C) } 102 \qquad \textbf{(D) } 103 \qquad \textbf{(E) } 104$

Proposed by jeteagle

Answer (E): By Heron's Formula, the area of the triangle is $\sqrt{s(s-a)(s-b)(s-c)}$, where s is the semiperimeter of the triangle, and a, b, and c are the side lengths. Therefore, the area is $\sqrt{70(70-49)(70-35)(70-56)} = 490\sqrt{3}$. Next, we see that $\frac{1}{2} \cdot AD \cdot BC = \frac{1}{2} \cdot BE \cdot AC = \frac{1}{2} \cdot CF \cdot AB = 490\sqrt{3}$, or the area of the triangle. Therefore, we can find that $AD = \frac{35\sqrt{3}}{2}$, $BE = 28\sqrt{3}$, and $CF = 20\sqrt{3}$. Now by the Pythagorean Theorem, we can calculate

$$AF = \sqrt{AC^2 - CF^2} = 5,$$

$$CE = \sqrt{BC^2 - BE^2} = 28,$$

$$BD = \sqrt{AB^2 - AD^2} = \frac{77}{2},$$

$$BF = AB - AF = 44,$$

$$CD = BC - BD = \frac{35}{2},$$
$$AE = AC - CE = 7.$$

We proceed with mass points. We see that $\frac{BD}{CD} = \frac{11}{5}$ so we give B a mass of 5 and C a mass of 11. Now, A will have a mass of 44 because $\frac{44}{5} = \frac{BF}{AF}$ where 44 and 5 are the masses of points A and B. Therefore, the mass of F will be $44 + 5 = 49$ and E will be $44 + 11 = 55$. Finally, we want to find $\frac{EG}{GF}$, which is just equal to the mass of F divided by the mass of E which is just $\frac{49}{55}$ so $m = 49$ and $n = 55$. Obviously 49 and 55 are relatively prime, so our answer is $49 + 55 = \boxed{\textbf{(E)}\ 104}$.

25. Define $s(k)$ as the period of the decimal expansion of $\frac{1}{k}$. Let S be the set of integers that are greater than 1 and can be written in the form $3^a \cdot 7^b$, where a and b are nonnegative integers. What is the value of $\frac{1}{s(k)}$ summed over all $k \in S$?

(A) $\frac{19}{6}$ **(B)** $\frac{229}{72}$ **(C)** $\frac{27}{8}$ **(D)** $\frac{11}{3}$ **(E)** $\frac{271}{72}$

Proposed by AIME12345

Answer (C): We will break the sum $\displaystyle\sum_{k \in S} \frac{1}{s(k)}$ into three pieces:

$$\mathcal{A} = \sum_{a=1}^{\infty} \frac{1}{s(3^a)}$$

$$\mathcal{B} = \sum_{b=1}^{\infty} \frac{1}{s(7^b)}$$

$$\mathcal{C} = \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \frac{1}{s(3^a \cdot 7^b)}$$

For a certain decimal, let d be the integer formed by the repeating block of the decimal. For example, in the decimal 0.121212...., $d = 12$. Then, we can write $\frac{1}{k}$ as such:

$$\frac{d}{10^{s(k)}} + \frac{d}{10^{2s(k)}} + \frac{d}{10^{3s(k)}} + \cdots = \frac{1}{k}.$$

The LHS is an infinite geometric series with a first term of $\frac{d}{10^{s(k)}}$ and a common ratio of $\frac{1}{10^{s(k)}}$. By the formula for the sum of an infinite geometric series, we get:

$$\frac{d}{10^{s(k)} - 1} = \frac{1}{k}.$$

We clear the denominators to get:

$$10^{s(k)} - 1 = dk.$$

Because d is an integer, we can take both sides of the equation in modulo k:

$$10^{s(k)} - 1 \equiv 0 \pmod{k}.$$

Clearly, $s(k)$ equals the smallest positive integer n for which $10^n - 1 \equiv 0 \pmod{k}$ holds.

First, we will consider the evaluation of $s(3^a)$.

Since 3^1 and 3^2 divide $10 - 1 = 9$ but not 3^3, $s(3^1) = 1$ and $s(3^2) = 1$.

Define $v_p(n)$ as the largest nonnegative integer m for which p^m divides n. Since 3 is divisible by $(10 - 1)$, we may apply Lifting the Exponent.

$$v_3(10^{s(k)} - 1) = v_3(9) + v_3(s(k)).$$

Since $k = 3^a$ and we are working in modulo 3^a, we want $v_3(10^{s(k)} - 1) = a$.

Therefore, using $v_3(10^{s(k)} - 1) = v_3(9) + v_3(s(k))$, we find that $a - 2 = v_3(s(k))$.

The smallest possible value $s(k)$ can equal here is 3^{a-2}. Therefore, for $a \geq 3$, $s(3^a) = 3^{a-2}$, but $s(3) = 1$ and $s(9) = 1$.

It immediately follows that:

$$\mathcal{A} = \sum_{a=1}^{\infty} \frac{1}{s(3^a)} = 1 + (1 + \frac{1}{3} + \frac{1}{9} + ...) = \frac{5}{2}.$$

Next, we will use a similar process to evaluate $s(7^b)$.

We can't apply Lifting the Exponent directly because $(10 - 1)$ is not divisible by 7. However, a quick check shows us that $n = 6$ is the smallest positive integer n for which $10^n - 1$ is divisible by 7. Let $t(7^b) = \frac{s(7^b)}{6}$.

$$10^{6t(7^b)} - 1 \equiv 0 \pmod{7^b}.$$

Now, we may apply Lifting the Exponent:

$$v_7(10^{6t(7^b)} - 1) = v_7(999999) + v_7(t(7^b)).$$

We want $v_7(10^{6t(7^b)} - 1) = b$ because we are working in modulo 7^b. In addition $v_7(999999) = 1$. Therefore,

$$b - 1 = v_7(t(7^b))$$
$$t(7^b) = 7^{b-1}$$
$$s(7^b) = 6 \cdot 7^{b-1}.$$

Therefore, $\mathcal{B} = \sum_{b=1}^{\infty} \frac{1}{s(7^b)} = \frac{1}{6}(1 + \frac{1}{7} + \frac{1}{49} + ...) = \frac{7}{36}$

Now, we will evaluate $s(3^a \cdot 7^b)$. Clearly $s(3^a \cdot 7^b) = \text{lcm}(s(3^a), s(7^b)) = \text{lcm}(s(3^a), 6 \cdot 7^{b-1})$.

We remember earlier that $s(3^1) = s(3^2) = 1$. However, for $a \geq 3$, $s(3^a) = 3^{a-2}$. When $a = 1$ or $a = 2$, $v_p(s(3^a))$ is always strictly less than $v_p(6 \cdot 7^{b-1})$. However, when $a \geq 3$, we have $v_p(s(3^a)) \geq v_p(6 \cdot 7^{b-1})$.

Therefore, for $a = 1$ and $a = 2$: $\sum_{b=1}^{\infty} \frac{1}{s(3^a \cdot 7^b)} = \frac{7}{36}$.

However, for $a \geq 3$: $\sum_{b=1}^{\infty} \frac{1}{s(3^a \cdot 7^b)} = \frac{7}{36 \cdot 3^{a-3}}$.

For our final sum:

$$\mathcal{A}+\mathcal{B}+\mathcal{C}=\frac{5}{2}+3\cdot\frac{7}{36}+\sum_{a=3}^{\infty}\sum_{b=1}^{\infty}\frac{1}{s(3^a\cdot 7^b)}$$

$$\frac{37}{12}+\sum_{a=3}^{\infty}\frac{7}{36\cdot 3^{a-3}}$$

$$\frac{37}{12}+\frac{7}{24}$$

$$\boxed{\textbf{(C) } \frac{27}{8}}.$$