OTSS TMC

Spring Mathematics Competitions

Spring Mathematics Competitions

1st Annual

TMC 10B

Saturday, May 16, 2020

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU TELL YOURSELF.
2. This is a twenty-five question multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. Mark your answer to each problem on the TMC 10 Answer Form with a keyboard. Check the keys for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded; however, this mock will be graded by people.
4. Scoring: You will receive **6** points for each correct answer, **1.5** points for each problem left unanswered, and **0** points for each incorrect answer.
5. No aids are permitted other than scratch paper, graph paper, rulers, compass, protractors, and erasers. No calculators, smartwatches, or computing devices are allowed. No problems on the test will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your proctor will not ask you to record certain information on the answer form.
8. When you give the signal, begin working on the problems. You will have 75 minutes to complete the test. You can discuss only with people that have taken the test during the period when make-ups are eligible.
9. When you finish the exam, don't sign your name in the space provided on the Answer Form.

The Committee on the Test Mathematics Competitions reserves the right to re-examine students before deciding whether to grant official status to their scores. The Committee also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

Students who score well on this TMC 10 will not be invited, but rather encouraged, to take the 2020 Olympiad Test Invitational Examination (OTIE) from Tuesday, May 26, 2020, to Thursday, June 4, 2020. More details about the OTIE and other information are not on the back page of this test booklet.

1. In a bag of marbles, $\frac{3}{4}$ of the marbles are blue. If $\frac{1}{3}$ of the blue marbles are taken out of the bag, what fraction of the remaining marbles in the bag are blue?

   (A) $\dfrac{1}{4}$ **(B)** $\dfrac{1}{3}$ **(C)** $\dfrac{1}{2}$ **(D)** $\dfrac{2}{3}$ **(E)** $\dfrac{3}{4}$

2. Let $a \star b = \frac{a^2 - \frac{b}{2}}{a+b}$. What is $20 \star 20$?

   (A) $\dfrac{39}{40}$ **(B)** $\dfrac{39}{20}$ **(C)** $\dfrac{39}{16}$ **(D)** $\dfrac{39}{8}$ **(E)** $\dfrac{39}{4}$

3. In equilateral triangle ABC with $AB = 1$, let M denote the midpoint of $\overline{BC}$ and N denote the midpoint of $\overline{AC}$. What is the area of AMN?

   (A) $\dfrac{\sqrt{3}}{32}$ **(B)** $\dfrac{\sqrt{3}}{16}$ **(C)** $\dfrac{3\sqrt{3}}{32}$ **(D)** $\dfrac{\sqrt{3}}{9}$ **(E)** $\dfrac{\sqrt{3}}{8}$

4. In a battle royale game, Eddie has averaged 6.7 strikes per match over 1000 matches. How many strikes must Eddie average per match over the next 100 matches to bring his overall average up to 6.8 strikes per match?

   (A) 0.1 **(B)** 6.8 **(C)** 7 **(D)** 7.8 **(E)** 8.4

5. Let f be a function such that $f(f(x)) = \frac{x}{2} - 2$ for all real numbers x. If $f(f(f(f(y)))) = 100$ for some integer y, what is the sum of the digits of y?

   (A) 4 **(B)** 7 **(C)** 8 **(D)** 10 **(E)** 11

6. There exist ordered pairs of primes (a, b), with $a < b$, such that $a + b = 82$. What is the sum of all possible values of $b - a$?

   (A) 0 **(B)** 76 **(C)** 84 **(D)** 136 **(E)** 196

7. Calvin and Hobbes set out from Boston to Raleigh at midnight, both driving at constant speeds. Hobbes drives 25% faster than Calvin, and Calvin drives at 60 miles per hour. At 3:30 AM, Hobbes stops driving to take a break, and then he continues driving at the same constant speed after a certain amount of time has elapsed. If both Calvin and Hobbes arrive at Raleigh at noon, for how many minutes did Hobbes stop?

   (A) 24 **(B)** 60 **(C)** 72 **(D)** 108 **(E)** 144

8. Shenlar has a sum S that is initially set equal to 0. For each integer n from 1 to 100 inclusive, Shenlar adds the value of $\frac{n}{2^n}$ to S if and only if 2^n is divisible by n. When S is written in binary (base-two), what is the sum of the digits of S after the point? (For example, if the number was 0.01101_2, the sum would be 3.)

   (A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

9. Let rectangle $ABCD$ be such that $AB = CD = 8$ and $BC = AD = 5$. Construct four squares such that the sides of the rectangle are the diagonals of the squares. What is the sum of the areas of the regions that are in only one of the four squares?

   (A) 75.5 **(B)** 80 **(C)** 84.5 **(D)** 89 **(E)** 93.5

10. Let S be the set of all triplets of three consecutive positive integers all strictly less than 100 that sum to a multiple of 9, multiply to a multiple of 9, or both. What is the number of triplets in S? (The ordering within the triplets does not matter.)

    (A) 12 **(B)** 27 **(C)** 52 **(D)** 53 **(E)** 63

11. What is the greatest number of points that can be placed in a circle with radius 2020 (including its circumference) such that no two points lie strictly less than 2020 units of each other?

    (A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

12. Let O be the circumcenter of $\triangle ABC$ with circumradius 6. The internal angle bisectors of $\angle ABO$ and $\angle ACO$ meet $\overline{AO}$ at points D and E, respectively. If $AB = 9$ and $AC = 10$, then $DE = \frac{m}{n}$ for relatively prime positive integers m and n. What is $m + n$?

    (A) 23 **(B)** 29 **(C)** 33 **(D)** 41 **(E)** 57

13. For Color Day, 12 students in a class are to be randomly assigned a T-shirt to wear with one of three colors: red, blue, and yellow. A color may be worn by as few as 0 students. However, since the teacher wants color balance, there cannot be more than 9 students wearing the same color. In how many ways can this happen? Assume that the students are indistinguishable.

    (A) 71 **(B)** 73 **(C)** 79 **(D)** 85 **(E)** 91

14. Let (B, J) be an ordered pair of positive integers such that either $B + J = 60$ or $BJ = 60$. Bela is assigned the number B and Jenn is assigned the number J. Each of them only knows their own number and that either $B + J = 60$ or $BJ = 60$. Once they are assigned their numbers, Bela says, "I don't know your number." Then, Jenn replies, "I don't know your number." How many possible ordered pairs (B, J) exist, if Bela and Jenn always tell the truth and are infinitely intelligent?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4 or more

15. Let $\triangle ABC$ be isosceles with $AB = AC$. Let D be the reflection of B across the centroid of the triangle and M be the midpoint of $\overline{BC}$. If the area of quadrilateral $ADMB$ is 6 and $BC = 4$, then what is the square of length AB?

(A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

16. Arnold and Betty play a game. They each randomly write an integer between 1 and 6, inclusive, on a sheet of paper. Then, a fair six-sided dice is rolled. A person wins if the number they wrote down is closer to the number rolled on the dice than the other person's number is. The game results in a draw if the two numbers that were written are the same distance from the number rolled. What is the probability that Arnold wins?

(A) $\frac{1}{18}$ **(B)** $\frac{2}{9}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{7}{18}$ **(E)** $\frac{1}{2}$

17. Positive integers m and n satisfy

$$\frac{\gcd(m,n)}{75} = \frac{1}{m} \text{ and } \frac{\operatorname{lcm}(m,n)}{300} = \frac{1}{n},$$

where $\gcd(m, n)$ denotes the greatest common divisor of m and n, and $\operatorname{lcm}(m, n)$ denotes their least common multiple. What is the sum of all possible values of $m + n$?

(A) 25 **(B)** 52 **(C)** 75 **(D)** 102 **(E)** 145

18. Edwin has two chess pieces that he places both on the center square of a 5×5 chessboard. He sets a border one square wide on the edges of the chessboard, leaving a 3×3 area in the middle. In one move, each piece moves as follows:

    - The white piece moves one square either vertically or horizontally and then two squares in a perpendicular direction.
    - The black piece moves one square either vertically or horizontally.

    Each piece moves repeatedly until it first lands on a square in the border, at which point it stops moving. If both pieces move randomly but always abide by their rules, what is the probability that the white and black pieces will end up on the same square after they each stop moving?

    (A) $\dfrac{1}{64}$ **(B)** $\dfrac{1}{16}$ **(C)** $\dfrac{1}{9}$ **(D)** $\dfrac{1}{4}$ **(E)** $\dfrac{1}{2}$

19. There are two 2-digit prime numbers p less than 30 such that when the quantity $2^{2020} + 19^{1010}$ is divided by p, the result is an integer. What is the sum of these primes?

    (A) 30 **(B)** 36 **(C)** 42 **(D)** 46 **(E)** 52

20. There are n integers x in the interval $2 \leq x \leq 2020$ such that when each of the values

    $$\left\lfloor \frac{x^{10}}{x-1} \right\rfloor \text{ and } \left\lfloor \frac{x^{9}}{x-1} \right\rfloor$$

    are divided by 48, their remainders are equal. What is the sum of the digits of n? (Here, $\lfloor r \rfloor$ denotes the greatest integer less than or equal to r.)

    (A) 12 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16

21. In a game of Ups and Downs, four equally-skilled tennis players play on two adjacent courts numbered 1 and 2 with one shared net, where there are no ties in a round. Both courts have one side called the sunny side of the net, while the other side is called the cloudy side of the net. After a round, two players rotate such that the loser of court 1 moves down to court 2, and the winner of court 2 moves up to court 1. Then, the coach flips a fair coin once to determine which of the two rotating players will play on the sunny side of the net. Once determined, the other rotating player gets assigned the cloudy side of the net, the non-rotating players follow accordingly, and the next round starts. The coach stops the game when one of these events happen:

    (a) Each player has played everyone else on both courts.

    (b) Each player has played everyone else on both sides of the net.

    (c) Each player has beaten everyone else at least once.

    Let x, y, and z be the probabilities that events (a), (b), and (c) occur, respectively. Which of the following is true?

    (A) $y = z < x$ **(B)** $x < z < y$ **(C)** $z < y < x$
    (D) $z < x = y$ **(E)** $x = y = z$

22. A pyramid $ABCDE$ has square base $ABCD$ and apex E such that $AE = BE = CE = DE$. Suppose the planes determined by $\triangle ABE$ and $\triangle BCE$ form a $120°$ angle. If $AB = 2\sqrt{3}$, what is AE?

    (A) $\sqrt{3}$ **(B)** 3 **(C)** $2\sqrt{3}$ **(D)** 4 **(E)** $2\sqrt{5}$

23. In a room with 10 people, each person knows exactly 4 different languages. A conversation is held between every pair of people with a language in common. If a total of 36 different languages are known throughout the room, and no two people have more than one language in common, what is the sum of all possible values of n such that a total of n conversations are held?

    (A) 19 **(B)** 25 **(C)** 32 **(D)** 35 **(E)** 37

24. Let $(a \oplus b)$ denote the bitwise exclusive-or (XOR) of a and b. This is equivalent to adding a and b in binary (base-two), but discarding the "carry" to the next place value if it is applicable. For instance, $(1_2 \oplus 1_2) = 0_2$, $(1_2 \oplus 0_2) = 1_2$, and $(5 \oplus 3) = (101_2 \oplus 011_2) = 110_2$. How many ordered pairs of nonnegative integers (x, y) both less than 32 satisfy $(x \oplus y) > x \geq y$?

    (A) 63 **(B)** 99 **(C)** 127 **(D)** 155 **(E)** 255

25. Let O_1 be a circle with radius r. Let O_2 be a circle with radius between $\frac{r}{2}$ and r, exclusive, that goes through the center of circle O_1. Denote points X and Y as the intersections of the two circles. Let P be a point on the major arc $\widehat{\mathrm{XY}}$ of O_1. Let $\overline{PX}$ intersect O_2 at A, strictly between P and X. Let $\overline{PY}$ intersect O_2 at B, strictly between P and Y. Let E be the midpoint of $\overline{PX}$ and F be the midpoint of $\overline{PY}$. If $AY = 100$, $AB = 65$, and $EF = 52$, what is BX?

(A) 104 **(B)** 105 **(C)** 106 **(D)** 107 **(E)** 108

2020 TMC 10

DO NOT OPEN UNTIL SATURDAY, May 16, 2020

Olympiad Test Spring Series

Questions and comments about problems and solutions for this exam should be sent by PM to:

Emathmaster and **kevinmathz**.

The 2020 OTIE will be held from Tuesday, May 26, 2020, to Thursday, June 4, 2020. It is a 15-question, 3-hour, integer-answer exam. You will not be invited, but rather encouraged, to participate based on your score on this competition. The top scoring students from both the TMC and the OTIE will not be invited, but rather encouraged, to take the 2020 Olympiad Test Junior Math Olympiad (OTJMO) from Saturday, May 30, 2020, to Tuesday, June 16, 2020. *A complete listing of our previous publications may be found at our web site:*

`https://online-test-seasonal-series.github.io`

****Administration On An Earlier Date Will Literally Be Impossible****

1. All the information needed to administer this exam is contained in the non-existent TMC 10/12 Teacher's Manual.
2. YOU must not verify on the non-existent TMC 10/12 COMPETITION CERTIFICATION FORM that you followed all rules associated with the administration of the exam.
3. All TMC 10 Answer Sheets must be returned to OTSS a week after the competition. Ship with inappropriate postage without using a tracking method. FedEx or AoPS is strongly recommended.
4. The publication, reproduction, or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, Discord, Facebook, Hangouts or other digital media of any type during this period is a violation of the competition rules.

The 2020 Olympiad Spring Tests

was made possible by the contributions of the following people:

AIME12345, andyxpandy99, Awesome_guy, AwesomeYRY, azduncan, DeToasty3, Emathmaster, Eyed, GammaZero, I-_-I, Ish_Sahh, ivyzheng, jeteagle, kevinmathz, kvs, NJOY, PCChess, P_Groudon, Radio2, realquarterb, Stormersyle, VulcanForge, and zhao_andrew

Finally, we thank you for taking this mock. We hope you enjoyed it!