2020 TMC 10B Problems and Solutions Document

Olympiad Test Spring Series

May 16, 2020 to May 25, 2020

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1. In a bag of marbles, $\frac{3}{4}$ of the marbles are blue. If $\frac{1}{3}$ of the blue marbles are taken out of the bag, what fraction of the remaining marbles in the bag are blue?

   (A) $\frac{1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{2}{3}$ **(E)** $\frac{3}{4}$

   Proposed by Emathmaster

   Answer (D): We have that $\frac{3}{4} \cdot \frac{1}{3} = \frac{1}{4}$ of the marbles are removed overall. This means $\frac{3}{4}$ of all the original marbles remain after taking out some of the blue ones. We also have that $\frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$ of the original marbles will remain in the bag and are blue. Therefore, the fraction of the remaining marbles that are blue is $\frac{1}{2} \div \frac{3}{4} = \boxed{\textbf{(D)}\ \dfrac{2}{3}}$.

2. Let $a \star b = \frac{a^2 - \frac{b}{2}}{a+b}$. What is $20 \star 20$?

   (A) $\frac{39}{40}$ **(B)** $\frac{39}{20}$ **(C)** $\frac{39}{16}$ **(D)** $\frac{39}{8}$ **(E)** $\frac{39}{4}$

   Proposed by PCChess

   Answer (E): Substituting in $a = 20$ and $b = 20$, our number is $\frac{20^2 - \frac{20}{2}}{40}$. This simplifies to $\boxed{\textbf{(E)}\ \dfrac{39}{4}}$

3. In equilateral triangle ABC with $AB = 1$, let M denote the midpoint of $\overline{BC}$ and N denote the midpoint of $\overline{AC}$. What is the area of AMN?

   (A) $\frac{\sqrt{3}}{32}$ **(B)** $\frac{\sqrt{3}}{16}$ **(C)** $\frac{3\sqrt{3}}{32}$ **(D)** $\frac{\sqrt{3}}{9}$ **(E)** $\frac{\sqrt{3}}{8}$

   Proposed by PCChess

Answer (B): Since M is the midpoint of $\overline{BC}$, the area of $\triangle CAM$ is half of $\triangle ABC$. Since N is the midpoint of $\overline{AC}$, the area of $\triangle AMN$ is half of $\triangle CAM$. Therefore, the area of $\triangle AMN$ is a quarter of $\triangle ABC$. We know that the area of $\triangle ABC$ is $\frac{\sqrt{3}}{4}$, so our answer is $\boxed{\textbf{(B) } \frac{\sqrt{3}}{16}}$.

4. In a battle royale game, Eddie has averaged 6.7 strikes per match over 1000 matches. How many strikes must Eddie average per match over the next 100 matches to bring his overall average up to 6.8 strikes per match?

(A) 0.1 **(B)** 6.8 **(C)** 7 **(D)** 7.8 **(E)** 8.4

Proposed by ivyzheng

Answer (D): After the 1000 matches, Eddie has made $6.7 \cdot 1000 = 6700$ strikes overall. If we wants to average 6.8 strikes per match after $1000 + 100 = 1100$, he must make $6.8 \cdot 1100 = 7480$ strikes overall after the 1100 matches. Therefore, he must make $7480 - 6700 = 780$ strikes in those 100 matches. The average is $\frac{780}{100} = \boxed{\textbf{(D) } 7.8}$

5. Let f be a function such that $f(f(x)) = \frac{x}{2} - 2$ for all real numbers x. If $f(f(f(f(y)))) = 100$ for some integer y, what is the sum of the digits of y?

(A) 4 **(B)** 7 **(C)** 8 **(D)** 10 **(E)** 11

Proposed by Emathmaster

Answer (B): Define $g(x) = f(f(x))$ for all real numbers x. Then, we have $g(x) = \frac{x}{2} - 2$ and $g(g(y)) = 100$. Evaluating the function once, we have $g(\frac{y}{2} - 2) = 100$. Evaluating the funtion again, we have $\frac{\frac{y}{2}-2}{2} - 2 = 100$. Solving for y, we get $y = 412$, so our answer is $\boxed{\textbf{(B) } 7}$.

6. There exist ordered pairs of primes (a, b), with $a < b$, such that $a + b = 82$. What is the sum of all possible values of $b - a$?

(A) 0 **(B)** 76 **(C)** 84 **(D)** 136 **(E)** 196

Proposed by ivyzheng

Answer (E): With $a < b$ and $a + b = 82$, we have $a \leq 40$. Clearly, $a \neq 2$ because $b = 80$ is not prime. We begin testing odd prime values of a up to 39 to get the following pairs: $(3, 79), (11, 71), (23, 59), (29, 53)$. We add up all the differences: $76 + 60 + 36 + 24 = \boxed{\textbf{(E) } 196}$.

7. Calvin and Hobbes set out from Boston to Raleigh at midnight, both driving at constant speeds. Hobbes drives 25% faster than Calvin, and Calvin drives at 60 miles per hour. At 3:30 AM, Hobbes stops driving to take a break, and then he continues driving at the same constant speed after a certain amount of time has elapsed. If both Calvin and Hobbes arrive at Raleigh at noon, for how many minutes did Hobbes stop?

(A) 24 **(B)** 60 **(C)** 72 **(D)** 108 **(E)** 144

Proposed by Awesome_guy

Answer (E): Since Calvin is driving at 60 miles per hour, we can conclude that Hobbes is driving at 75 miles per hour. Additionally, Calvin drives for 12 hours nonstop, so the distance of the trip is $60 \cdot 12 = 720$. Therefore, Hobbes spends $\frac{720}{75} = 9.6$ hours driving and $12 - 9.6 = 2.4$ hours stopped. Converting this into minutes, we see that Hobbes stopped for $\boxed{\textbf{(E)}\ 144}$ minutes.

8. Shenlar has a sum S that is initially set equal to 0. For each integer n from 1 to 100 inclusive, Shenlar adds the value of $\frac{n}{2^n}$ to S if and only if 2^n is divisible by n. When S is written in binary (base-two), what is the sum of the digits of S after the point? (For example, if the number was 0.01101_2, the sum would be 3.)

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Proposed by Radio2

Answer (A): Notice that for 2^n to be divisible by n, the prime factorization of n must consist solely of twos. Therefore, the only possible values of n are $1, 2, 4, 8, 16, 32,$ or 64. For each value of n, 2^n is clearly greater than n, which means that the value added to S will be less than one. Also, each value of $\frac{n}{2^n}$ is unique except the cases where $n = 1$ and $n = 2$. In that case, those possible values sum to one. Since there are 7 powers of 2 that are added to S, with one overlap, the value of S will have 6 ones. However, one of the ones will be before the radix point, so the answer is $\boxed{\textbf{(A)}\ 5}$.

9. Let rectangle $ABCD$ be such that $AB = CD = 8$ and $BC = AD = 5$. Construct four squares such that the sides of the rectangle are the diagonals of the squares. What is the sum of the areas of the regions that are in only one of the four squares?

(A) 75.5 **(B)** 80 **(C)** 84.5 **(D)** 89 **(E)** 93.5

Proposed by jeteagle

Answer (B): We see that the overlapping area is a square. We let the length of the diagonal of the square be x. We have $x = 4+4-5 = 3$, so the area of the small square is 4.5. The area of the 4 squares not counting overlaps is $8^2 + 5^2 = 89$. But since this counts the overlapped square twice, and we want to find the area of the non-overlapping parts, we have the answer be $89 - 2 \cdot 4.5 = \boxed{\textbf{(B)}\ 80}$.

10. Let S be the set of all triplets of three consecutive positive integers all strictly less than 100 that sum to a multiple of 9, multiply to a multiple of 9, or both. What is the number of triplets in S? (The ordering within the triplets does not matter.)

(A) 12 **(B)** 27 **(C)** 52 **(D)** 53 **(E)** 63

Proposed by ivyzheng

Answer (D): We use PIE based on the middle number of the triplet to keep track:

Adding: the middle number must be $3, 6, 0 \mod 9 \implies 3, 6, 9 \ldots 96 = 32$ triplets

Multiplying: the middle number must be $8, 0, 1 \mod 9 \implies 8, 9, 10 \ldots 89, 90, 91 \ldots 98 = 3 \cdot 10 + 1 = 31$ triplets

Both: in both, the middle number being $0 \mod 9$ causes overlap so $0 \mod 9 \implies 9, 18, \ldots 90 = 10$ triplets.

Using PIE, we have $32 + 31 - 10 = \boxed{\textbf{(D) } 53}$.

11. What is the greatest number of points that can be placed in a circle with radius 2020 (including its circumference) such that no two points lie strictly less than 2020 units of each other?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Proposed by Awesome_guy

Answer (C): In order to maximize the amount of points, we want to space them out as much as possible. Therefore, we want to place as many as we can on the circumference. First, note that the length of the circumference is $2 \cdot 2020\pi = 4040\pi \approx 12,692$. Since the points must be at least 2020 units away from each other, we can place 6 points on the circumference. Finally, we can chose the point in the center, so we have a total of $\boxed{\textbf{(C) } 7}$ points.

12. Let O be the circumcenter of $\triangle ABC$ with circumradius 6. The internal angle bisectors of $\angle ABO$ and $\angle ACO$ meet $\overline{AO}$ at points D and E, respectively. If $AB = 9$ and $AC = 10$, then $DE = \frac{m}{n}$ for relatively prime positive integers m and n. What is $m + n$?

(A) 23 **(B)** 29 **(C)** 33 **(D)** 41 **(E)** 57

Proposed by Emathmaster and Ish_Sahh

Answer (A): By the Angle Bisector Theorem on $\triangle ABO$, $\frac{AD}{DO} = \frac{AB}{BO} = \frac{3}{2}$. Because $AO = 6$, we have $DO = \frac{12}{5}$. By the Angle Bisector Theorem on $\triangle ACO$, $\frac{OE}{EA} = \frac{CO}{CA} = \frac{3}{5}$. Therefore, $AE = \frac{15}{4}$. We also know that $DE = |AO - OD - AE|$. Plugging in the numbers, we get $DE = \frac{3}{20}$. The answer is $\boxed{\textbf{(A) } 23}$.

13. For Color Day, 12 students in a class are to be randomly assigned a T-shirt to wear with one of three colors: red, blue, and yellow. A color may be worn by as few as 0 students. However, since the teacher wants color balance, there cannot be more than 9 students wearing the same color. In how many ways can this happen? Assume that the students are indistinguishable.

(A) 71 **(B)** 73 **(C)** 79 **(D)** 85 **(E)** 91

Proposed by DeToasty3

Answer (B): Use complementary counting.

Case 1: One color has 12 students. There are 3 colors, each with 1 way, for a total of 3 ways.

Case 2: One color has 11 students. We choose 1 color of 3 to have 11 students, and 1 other color to have 1 student, for a total of 6 ways.

Case 3: One color has 10 students. We choose 1 color of 3 to have 10 students. Now, either 1 other color has 2 students, or both of the other colors have 1 student each, for a total of $6 + 3 = 9$.

The total number of ways without restrictions is, by stars and bars, $\binom{14}{2} = 91$, so the total number of ways is $91 - 3 - 6 - 9 = \boxed{\textbf{(B) } 73}$.

14. Let (B, J) be an ordered pair of positive integers such that either $B + J = 60$ or $BJ = 60$. Bela is assigned the number B and Jenn is assigned the number J. Each of them only knows their own number and that either $B + J = 60$ or $BJ = 60$. Once they are assigned their numbers, Bela says, "I don't know your number." Then, Jenn replies, "I don't know your number." How many possible ordered pairs (B, J) exist, if Bela and Jenn always tell the truth and are infinitely intelligent?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4 or more

Proposed by kevinmathz

Answer (C): First, if Bela says she doesn't know Jenn's number, then her number must be a factor of 60 and not equal to 60. Now, Jenn knows that Bela's number must be a factor of 60 not equal to 60 so if she doesn't know Bela's number, then her number is also a factor of 60 and not equal to 60. Now here's the catch: Jenn knows Bela's number is a factor of 60 and not equal to 60 so as a result, her number must be 30 or she would know that it's definitely the product. With Jenn's number as 30, Bela's number is 2 or 30 so our answer is $\boxed{\textbf{(C) } 2}$.

15. Let $\triangle ABC$ be isosceles with $AB = AC$. Let D be the reflection of B across the centroid of the triangle and M be the midpoint of $\overline{BC}$. If the area of quadrilateral $ADMB$ is 6 and $BC = 4$, then what is the square of length AB?

(A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

Proposed by PCChess and DeToasty3

Answer (D): Let G be the centroid of $\triangle ABC$. Because D is the reflection of B over G, we have $BG = GD$. Therefore, $[ABG] = [AGD]$ and $[BGM] = [GMD]$. It then follows that $[ABM] = [AMD]$. We are given that $[ADMB] = 6$, so $[ABM] = 3$. Because $BM = 2$, it follows that $AM = 3$. By the Pythagorean Theorem $AB^2 = 3^2 + 2^2 = \boxed{\textbf{(D) } 13}$.

16. Arnold and Betty play a game. They each randomly write an integer between 1 and 6, inclusive, on a sheet of paper. Then, a fair six-sided dice is rolled. A person wins if the number they wrote down is closer to the number rolled on the dice than the other person's number is. The game results in a draw if the two numbers that were written are the same distance from the number rolled. What is the probability that Arnold wins?

(A) $\frac{1}{18}$ **(B)** $\frac{2}{9}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{7}{18}$ **(E)** $\frac{1}{2}$

Proposed by PCChess

Answer (D): We can simply solve for the probability of a draw and the subtract from 1 and divide by 2.

Case 1 - They both write the same number: This obviously results in a draw. There is a $\frac{1}{6}$ chance that they write the same number.

Case 2 - The difference between the 2 numbers is 2: There are 4 possibilities. $(1,3),(2,4),(3,5),(4,6)$. Each time, the roll must be the number between the 2 numbers written down. Hence, for each possibility, the probability that it occurs is $\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}=\frac{1}{216}$. There are 8 possibilities, so we multiply by 8 to get $\frac{8}{216}=\frac{1}{27}$.

Case 3 - The difference between the 2 numbers is 4: There are 2 possibilities. $(1,5),(2,6)$. The roll must be in the middle of the 2 numbers. The probability is $4\cdot\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}=\frac{1}{54}$.

Summing, the probability of a draw is $\frac{1}{6}+\frac{1}{27}+\frac{1}{54}=\frac{12}{54}=\frac{2}{9}$. The probability that Arnold wins is therefore $\boxed{\textbf{(D) } \frac{7}{18}}$.

17. Positive integers m and n satisfy

$$\frac{\gcd(m,n)}{75}=\frac{1}{m} \text{ and } \frac{\operatorname{lcm}(m,n)}{300}=\frac{1}{n},$$

where $\gcd(m,n)$ denotes the greatest common divisor of m and n, and $\operatorname{lcm}(m,n)$ denotes their least common multiple. What is the sum of all possible values of $m+n$?

(A) 25 **(B)** 52 **(C)** 75 **(D)** 102 **(E)** 145

Proposed by PCChess

Answer (D): Note that since $\gcd(m,n)\cdot\operatorname{lcm}(m,n)=mn$, multiplying the two equations yields

$$\frac{mn}{22500}=\frac{1}{mn}.$$

Then, $mn=150$. We have the following possible solutions (and with the numbers flipped):

$$(1,150),(2,75),(3,50),(5,30),(6,25),(10,15).$$

Checking them, it is easy to see that only $(m=75,n=2)$ and $(m=15,n=10)$ work. Therefore, the answer is $77+25=\boxed{\textbf{(D) } 102}$.

18. Edwin has two chess pieces that he places both on the center square of a 5×5 chessboard. He sets a border one square wide on the edges of the chessboard, leaving a 3×3 area in the middle. In one move, each piece moves as follows:

- The white piece moves one square either vertically or horizontally and then two squares in a perpendicular direction.
- The black piece moves one square either vertically or horizontally.

Each piece moves repeatedly until it first lands on a square in the border, at which point it stops moving. If both pieces move randomly but always abide by their rules, what is the probability that the white and black pieces will end up on the same square after they each stop moving?

(A) $\frac{1}{64}$ **(B)** $\frac{1}{16}$ **(C)** $\frac{1}{9}$ **(D)** $\frac{1}{4}$ **(E)** $\frac{1}{2}$

Proposed by Radio2

Answer (B): We assign coordinates, with the center square at $(0, 0)$.

We see that the white piece immediately moves onto the border, onto one of the eight squares $(\pm 2, \pm 1)$ or $(\pm 1, \pm 2)$, with the probability of being on any one of these squares being $1/8$. We must thus compute the probability that the black piece ends on one of these squares.

Exploiting symmetry, we see that the probabilities of the black piece ending on: $(2, 0)$, $(-2, 0)$, $(0, 2)$, $(0, -2)$ are equal, as are the probabilities of ending on these eight squares: $(\pm 2, \pm 1)$ or $(\pm 1, \pm 2)$.

We see that no matter what, after the first move the black piece is adjacent to one of the four squares $(2, 0), (-2, 0), (0, 2), (0, -2)$. Now let the probability of ending on one of these four squares be P. We see that there is a $1/4$ chance the black piece moves directly onto one of these squares, a $1/4$ chance it moves back to the origin, and a $1/2$ chance it moves to $(\pm 1, \pm 1)$. From the origin, the piece's only move is back to a square adjacent to one of the four, so this case's probability is $P/4$. From $(\pm 1, \pm 1)$ there is a $1/2$ chance the piece goes onto a border square and a $1/2$ chance it returns to a square adjacent to one of the four. So this case's probability is $(1/2)(1/2)P = P/4$.

Thus $P = 1/4 + P/4 + P/4$, so $P = 1/2$. Thus the probability of ending on a square the white piece can also end on is $1 - 1/2 = 1/2$, and the probability both pieces land on the same square is thus $\boxed{\textbf{(B)}\ \frac{1}{16}}$.

19. There are two 2-digit prime numbers p less than 30 such that when the quantity $2^{2020} + 19^{1010}$ is divided by p, the result is an integer. What is the sum of these primes?

(A) 30 **(B)** 36 **(C)** 42 **(D)** 46 **(E)** 52

Proposed by andyxpandy99

Answer (C): Observe that $2^{2020} + 19^{1010} = 16^{505} + 361^{505}$. Note that we can factor the RHS as

$$(16 + 361)(16^{504} - 16^{503} \cdot 361 + \cdots - 16 \cdot 361^{504} + 361^{505}).$$

$16 + 361 = 377 = 13 \cdot 29$. Now we know that 13 and 29 divide $2^{2020} + 19^{1010}$. We are given that there are two 2-digit primes less than 30 that divide $2^{2020} + 19^{1010}$, so our two primes must be 13 and 29. Our answer is $\boxed{\textbf{(C)}\ 42}$.

20. There are n integers x in the interval $2 \le x \le 2020$ such that when each of the values

$$\left\lfloor \frac{x^{10}}{x-1} \right\rfloor \text{ and } \left\lfloor \frac{x^9}{x-1} \right\rfloor$$

are divided by 48, their remainders are equal. What is the sum of the digits of n? (Here, $\lfloor r \rfloor$ denotes the greatest integer less than or equal to r.)

(A) 12 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16

Proposed by Ish_Sahh

Answer (A): This equation can be rewritten as $\lfloor \frac{x^{10}-1}{x-1} + \frac{1}{x-1} \rfloor \equiv \lfloor \frac{x^9-1}{x-1} + \frac{1}{x-1} \rfloor \pmod{48}$. Since $x-1$ is a factor of $x^{10}-1$ and x^9-1, both sides can be expanded to $\lfloor x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1+\frac{1}{x-1} \rfloor \equiv \lfloor x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1+\frac{1}{x-1} \rfloor \pmod{48}$. Each of the terms x^n for integral x, n are integers, so they can be brought outside of the floor function. This would make the equation $x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1+\lfloor \frac{1}{x-1} \rfloor \equiv x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1+\lfloor \frac{1}{x-1} \rfloor \pmod{48}$. After like terms are cancelled out, all that's left is $x^9 \equiv 0 \pmod{48}$, which says that x^9 is divisible by 48. Since $48 = 2^4 \cdot 3^1$, all that is needed is x to be divisible by 6. With $1 < x < 2020$, there are $\lfloor \frac{2020}{6} \rfloor - \lfloor \frac{1}{6} \rfloor = 336$ values for x that are divisible by 6 and satisfy the equation. The digit sum of 336 gives the answer of $\boxed{\textbf{(A) } 12}$.

21. In a game of Ups and Downs, four equally-skilled tennis players play on two adjacent courts numbered 1 and 2 with one shared net, where there are no ties in a round. Both courts have one side called the sunny side of the net, while the other side is called the cloudy side of the net. After a round, two players rotate such that the loser of court 1 moves down to court 2, and the winner of court 2 moves up to court 1. Then, the coach flips a fair coin once to determine which of the two rotating players will play on the sunny side of the net. Once determined, the other rotating player gets assigned the cloudy side of the net, the non-rotating players follow accordingly, and the next round starts. The coach stops the game when one of these events happen:

(a) Each player has played everyone else on both courts.
(b) Each player has played everyone else on both sides of the net.
(c) Each player has beaten everyone else at least once.

Let x, y, and z be the probabilities that events (a), (b), and (c) occur, respectively. Which of the following is true?

(A) $y = z < x$ **(B)** $x < z < y$ **(C)** $z < y < x$ **(D)** $z < x = y$ **(E)** $x = y = z$

Proposed by AIME12345

Answer (A): The intuition is that there must be 12 mini-events for any of the three events to occur, i.e. 12 ways for all 6 pairs of players to play on both courts, 12 ways for all 6 pairs of players to play on both sides of either court, and 12 ways for all 6 pairs of players to beat each other once. So for each round, we want to add 0, 1, or 2 mini-events towards the total number of 12. Event A is the most likely to occur, because we can only add 0 or 2 mini-events towards the total each round, as two players who played on court 1 for the first time implies that the other two players played on court 2 for the first time. Events B and C are allowed to add 1 mini-events towards the total. Although it may not seem like it, these two events form a bijection, as we can say a player playing on the sun side of the court is equivalent to a win. Because every player has an equal chance of rotating after a round, the coach's one coin flip is actually equivalent to forming "win-loss" results for both courts! Therefore, our answer is $\boxed{\textbf{(A)}\ y = z < x}$.

22. A pyramid $ABCDE$ has square base $ABCD$ and apex E such that $AE = BE = CE = DE$. Suppose the planes determined by $\triangle ABE$ and $\triangle BCE$ form a 120° angle. If $AB = 2\sqrt{3}$, what is AE?

    (A) $\sqrt{3}$ **(B)** 3 **(C)** $2\sqrt{3}$ **(D)** 4 **(E)** $2\sqrt{5}$

    Proposed by P_Groudon

    Answer (B): In the plane of face $\triangle ABE$, let the foot of the altitude from A to BE be F. By symmetry, the foot of the altitude from C to BE along the plane of face $\triangle BCE$ is also F. In addition $AF = FC$. Because the planes form a 120° angle, $\triangle AFC$ is an isosceles triangle with $\angle AFC = 120^\circ$. Because AC is the diagonal of square $ABCD$ with $AB = 2\sqrt{3}$, we have $AC = 2\sqrt{6}$. By the 120-30-30 triangle, $AF = 2\sqrt{2}$. By Pythagorean Theorem on $\triangle AFB$, we have $FB = 2$. Let $AE = x$. Then, $EF = x - 2$. By Pythagorean Theorem on $\triangle AEF$, we have $(x-2)^2 + 8 = x^2$, solving for x, we find $x = 3$, so our answer is $\boxed{\textbf{(B)}\ 3}$.

23. In a room with 10 people, each person knows exactly 4 different languages. A conversation is held between every pair of people with a language in common. If a total of 36 different languages are known throughout the room, and no two people have more than one language in common, what is the sum of all possible values of n such that a total of n conversations are held?

    (A) 19 **(B)** 25 **(C)** 32 **(D)** 35 **(E)** 37

    Proposed by DeToasty3

    Answer (C): We have that 10 people speak 4 languages. Label each of the $10 \cdot 4 = 40$ languages associated with a person with a token. Since there are 36 languages, each of the languages must have at least one token. After we give each language a token, we have $40 - 36 = 4$ remaining tokens left to distribute. Since no two people have more than one language in common, we may assume that for any language with k people speaking it, where $k \geq 2$, $\binom{k}{2}$ is the number of conversations held, with no need to subtract "extra" conversations for if two people had conversations in two or more different languages. If $k = 1$, then no conversations take place. Our goal is to find all of the cases that result after distributing 4 tokens into 36 languages, where the languages are indistinguishable, and see how many conversations happen in each case.

    Case 1: One language gets all four tokens. Then, a total of $4 + 1 = 5$ tokens are in that language, giving us $\binom{5}{2} = 10$ conversations.

Case 2: One language gets three tokens, and one other language gets one token. Then, a total of $3+1=4$ tokens are in one language, and a total of $1+1=2$ tokens are in the other language, giving us $\binom{4}{2}+\binom{2}{2}=6+1=7$ conversations.

Case 3: Two languages get two tokens each. Then, a total of $2+1=3$ tokens are in each language, giving us $2\binom{3}{2}=6$ conversations.

Case 4: One language gets two tokens, and two other languages get one token each. Then, a total of $2+1=3$ tokens are in one language, and a total of $1+1=2$ tokens are in the other two languages, giving us $\binom{3}{2}+2\binom{2}{2}=3+2=5$ conversations.

Case 5: Four languages get one token each. Then, a total of $1+1=2$ tokens are in the four languages, giving us $4\binom{2}{2}=4$ conversations.

Note that in each case, it is possible to assign each token to one of the ten people such that the problem's conditions are satisfied, so the sum of all possible values of n is $10+7+6+5+4=$ $\boxed{\textbf{(C) } 32}$.

24. Let $(a \oplus b)$ denote the bitwise exclusive-or (XOR) of a and b. This is equivalent to adding a and b in binary (base-two), but discarding the ”carry” to the next place value if it is applicable. For instance, $(1_2 \oplus 1_2)=0_2$, $(1_2 \oplus 0_2)=1_2$, and $(5 \oplus 3)=(101_2 \oplus 011_2)=110_2$. How many ordered pairs of nonnegative integers (x, y) both less than 32 satisfy $(x \oplus y) > x \geq y$?

(A) 63 **(B)** 99 **(C)** 127 **(D)** 155 **(E)** 255

Proposed by PCChess

Answer (D): First, we notice that when we apply $x \oplus y$ for binary numbers x and y, the addition in one place has no effect on the addition in other places because there are no carries. Clearly, x and y cannot have the same number of digits in binary. This is because if they did have the same number of digits (when leading zeroes are disregarded), both would have a leading 1 and then $x \oplus y$ would have a leading 0. This would cause $x \oplus y$ to be less than x, which is bad.

It follows that y must have fewer digits than x in binary. To make the following claim, we will disregard leading zeroes in x and y. We know y must have a leading 1. Suppose that leading 1 is in the ds place. If x has a 1 in the ds place. Then, $x \oplus y$ will have a 0 in the ds place. Because everything to the left of the ds place will be the same between x and $x \oplus y$, this causes $x > x \oplus y$.

However, if x has a 0 in the ds place, $x \oplus y$ will have a 1 in the ds place. Again, because everything to the left of the ds place will be the same between x and $x \oplus y$, this causes $x \oplus y > x$.

We will do casework based on the number of digits of y when y is written in binary. Now we will regard leading 0s only for x. Note that x can have up to 5 digits.

Case 1: y has 4 digits

That means x looks like:

_0___

On the other hand, y looks like:

1___

Note that the digit to the left of 0 in x must be a 1. We may assign whatever we want to the remaining 6 blank spaces between x and y because the addition in those places have no effect on the addition in the other places. We have $2^6 = 64$ cases here.

Case 2: y has 3 digits

That means x looks like:

__0__

On the other hand, y looks like:

1__

Consider the two blank spaces to the left of the 0 in x. We can opt for leading zeroes. For example, we can have 010__, which is the equivalent to 10__ while still satisfying the condition. However, the two blank spaces cannot be both zero. So we have $2^2 - 1$ ways to fill in those two blank spaces. Again, we can do whatever we want with the remaining 4 blank spaces. So we have $3 \cdot 2^4 = 48$ cases here.

Case 3: y has 2 digits

x : ___0_

y : 1_

Again, the three blank spaces to the left of the zero in x can't ALL be 0. So we have $2^3 - 1$ ways to fill in those spaces. We can do whatever we want with the other two spaces. So we have $7 \cdot 2^2 = 28$ cases here.

Case 4: y has 1 digit

x : ____0

y: 1

The four blank spaces in x cannot all be 0, so we have 15 cases here.

We have $\boxed{\textbf{(D) } 155}$ cases total.

25. Let O_1 be a circle with radius r. Let O_2 be a circle with radius between $\frac{r}{2}$ and r, exclusive, that goes through the center of circle O_1. Denote points X and Y as the intersections of the two circles. Let P be a point on the major arc $\widehat{XY}$ of O_1. Let $\overline{PX}$ intersect O_2 at A, strictly between P and X. Let $\overline{PY}$ intersect O_2 at B, strictly between P and Y. Let E be the midpoint of $\overline{PX}$ and F be the midpoint of $\overline{PY}$. If $AY = 100$, $AB = 65$, and $EF = 52$, what is BX?

(A) 104 **(B)** 105 **(C)** 106 **(D)** 107 **(E)** 108

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Answer (B): Denote C_1 as the center of O_1. Because C_1AXY is cyclic and PC_1X is isosceles with $PC_1 = C_1X$, we have $\angle C_1YA = \angle C_1XA = \angle XPC_1 = \angle APC_1$. Using $\angle C_1YA = \angle APC_1$ and $PC_1 = C_1Y$, it follows that $PA = AY$, so AC_1 is perpendicular to PY. Because PY is a chord of the circle and AC_1 passes through the center of the circle, AC_1 must pass through F. Similarly, BC_1 must pass through E.

Therefore, we have that $\triangle PAB$ has orthocenter C_1. From $\triangle PBX$ having $PB = BX$ and $\triangle PAY$ having $PA = AY$, we are really after PB. We are given that $AY = PA = 100$.

Clearly, $\triangle PEF \simeq \triangle PBA$, so $\frac{PF}{PA} = \frac{EF}{AB}$. Plugging in $PA = 100$, $AB = 65$, and $EF = 52$, this tells us that $PF = 80$. Through Pythagorean Theorem on $\triangle APF$, we get $AF = 60$. Then, by Pythagorean Theorem on $\triangle AFB$, we get $FB = 25$. Therefore, $PB = BX = \boxed{\textbf{(B) } 105}$.