2020 TMC 12A Problems and Solutions Document

Olympiad Test Spring Series

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1. There are 9 large monkeys and 10 little monkeys who want some bananas. Each little monkey needs 1 banana to be full, while each large monkey needs 2 bananas to be full. Given that there are 15 bananas, what is the maximum amount of monkeys that can become full?

   (A) 8 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 13

   Proposed by PCChess

   Answer (D): The little monkeys take less space so we first use the 10 of them up. Afterwards, we have 5 bananas, 2 monkeys so our answer is $10 + 2 = \boxed{\textbf{(D)}\ 12}$.

2. The three digit number $5\underline{A}2$ is divisible by 4. What is the sum of the possible values of the digit A?

   (A) 10 **(B)** 14 **(C)** 15 **(D)** 20 **(E)** 25

   Proposed by Emathmaster

   Answer (E): By the divisibility by 4 rule, the number formed by the last two digits of $5\underline{A}2$ must be divisible by 4, so we can disregard the 5 in the hundreds place.

   Clearly 02 is not divisible by 4, but 12 is. We notice that 10 is not divisible by 4, but 20 is, so $12 + 20 \cdot k$ for some nonnegative integer k will be divisible by 4. It then follows that A must be odd, so A can be $1, 3, 5, 7,$ or 9. Our answer is $\boxed{\textbf{(E)}\ 25}$.

3. What is the value of

$$\left(3+2\sqrt{2}\right) + \left(\frac{1}{3-2\sqrt{2}}\right) + \left(3-2\sqrt{2}\right) + \left(\frac{1}{3+2\sqrt{2}}\right)?$$

   (A) $4\sqrt{2}$ **(B)** 6 **(C)** $8\sqrt{2}$ **(D)** $6+4\sqrt{2}$ **(E)** 12

Proposed by Emathmaster

Answer (E): To remove the radicals from the denominator, we multiply the denominator and numerator by the conjugate:

$$\left(\frac{1}{3-2\sqrt{2}}\right)\cdot\left(\frac{3+2\sqrt{2}}{3+2\sqrt{2}}\right)=3+2\sqrt{2}$$

Similarly,

$$\left(\frac{1}{3+2\sqrt{2}}\right)=3-2\sqrt{2}$$

Adding everything up gives us $\boxed{\textbf{(E) } 12}$.

4. On a ten-question True/False test, Neel only knows the answer to three of the questions! As a result, he flips a fair coin to determine his answers for the rest of the questions. Given that a passing grade is anything above 60% in Neel's school and that he correctly answers all three questions where he knows the answer to, what is the probability that Neel passes the test?

   (A) $\frac{11}{64}$ **(B)** $\frac{29}{128}$ **(C)** $\frac{193}{512}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{99}{128}$

   Proposed by kevinmathz

   Answer (D): Neel needs at least 4 of the last 7 because a passing grade is above 60%. Thus, by symmetry, getting 3 and 4 are symmetric, and so are 2 and 5, 1 and 6, and 0 and 7, so the probability Neel passes the test is $\boxed{\textbf{(D) } \frac{1}{2}}$.

5. Let ABC be an equilateral triangle. Next, let D be on the extension of $\overline{BC}$ past point B such that $\angle BAD = 30^\circ$, and let E be on the extension of $\overline{BC}$ past point C such that $\angle EAC = 30^\circ$. If $BC = 2$, what is the area of DAE?

   (A) $2\sqrt{2}$ **(B)** 3 **(C)** $2\sqrt{3}$ **(D)** $3\sqrt{3}$ **(E)** 6

   Proposed by Ish_Sahh

   Answer (D): We see that the base of $\triangle DAE$ has length 6 and the height has length $\sqrt{3}$. Thus, the area is $\frac{1}{2}\cdot 6\cdot\sqrt{3}=\boxed{\textbf{(D) } 3\sqrt{3}}$.

6. Call an ordered pair of positive primes (a, b) *cool* if $a = b - 10$. Suppose that for some integer n, there exists a list of primes $P_1, P_2, \ldots, P_n$ such that (P_i, P_{i+1}) is *cool* for all $1 \le i \le n-1$. What is the largest possible value of n?

   (A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 10

   Proposed by PCChess

Answer (B): If $n \geq 3$, it is guaranteed that at least one number is divisible by 3. This means that n can be maximized by making $P_1 = 3$. Checking, the list of numbers $3, 13, 23$ works, so the answer is $\boxed{\textbf{(B) } 3}$.

7. At Lexington High School, it is customary for people to not use adjacent stalls in a bathroom. Some (possibly empty) subset of five different kids want to use a row of four stalls at the same time. In how many ways can they do so?

(A) 53 **(B)** 57 **(C)** 81 **(D)** 93 **(E)** 141

Proposed by Emathmaster

Answer (C): We do casework based on how many stalls are occupied.

Case 1: No stalls occupied

There is 1 case here.

Case 2: 1 stall occupied

We may choose any of the 5 students and any of the 4 stalls. So we have $5 \cdot 4 = 20$ cases here.

Case 3: 2 stalls occupied

In order for no two adjacent stalls to be occupied, we can either have stalls 1 and 3 occupied, stalls 2 and 4 occupied, or stalls 1 and 4 occupied. This gives 3 different acceptable configurations. We may choose any of the 5 students to go in the left stall and then any 4 of the remaining students in the other stall. This gives $3 \cdot 5 \cdot 4 = 60$ cases here.

Case 4: 3 stalls occupied

This would require some two adjacent stalls, so we cannot have 3 stalls occupied.

In total, this gives us $1 + 20 + 60 = \boxed{\textbf{(C) } 81}$ ways.

8. Let x be a positive integer such that

$$i^{x^4-3x^3+5x^2+7x-11} = i^{x^3+2x^2+5x+3},$$

where $i = \sqrt{-1}$. Find the set of all possible remainders when x is divided by 4.

(A) $\emptyset$ **(B)** $\{1\}$ **(C)** $\{0,1\}$ **(D)** $\{0,3\}$ **(E)** $\{1,3\}$

Proposed by jeteagle

Answer (E): For $i^{x^4-3x^3+5x^2+7x-11} = i^{x^3+2x^2+5x+3}$, we require the exponents to be congruent in modulo 4.

Therefore, we want $x^4 - 3x^3 + 5x^2 + 7x - 11 \equiv x^3 + 2x^2 + 5x + 3 \pmod 4$.

Moving all the terms to one side and taking mod 4 of the coefficients:

$$x^4 - 4x^3 + 3x^2 + 2x - 14 \equiv 0 \pmod 4$$

$$x^4 - x^2 + 2x + 2 \equiv 0 \pmod 4$$

We notice that if x is even, each of the terms on the left hand side will be divisible by 4 except the 2. Therefore, if x is even, we get $2 \equiv 0 \pmod 4$, which is a contradiction.

Now, suppose x is odd, which means $x \equiv \pm 1 \pmod 4$. Then x to even power is congruent to $1 \pmod 4$, so the x^4 and x^2 cancel each other out. We can check that if we plug in $x \equiv \pm 1 \pmod 4$ to $2x + 2 \pmod 4$, both values work. Therefore, the set of remainders is $\boxed{\textbf{(E) } \{1, 3\}}$.

9. To celebrate Bela's birthday, Jenn decides to make a cake in the shape of a right cylinder with a radius of 2 and a height of 10. Strangely, Jenn covers the entire outside (including the bottom) of the cake with frosting and cuts the cake such that each cut is parallel to the base of the cake, and each resulting slice is a cylinder. There is only sponge and no frosting on the inside of the cake. On each slice, Jenn wants the amount of frosting to be the same. If Jenn cuts the cake into 8 parts, what is the height of the slice that contains the top of the cake? (Assume the frosting has negligible thickness.)

   (A) $\frac{1}{4}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{2}{3}$ **(D)** 1 **(E)** $\frac{6}{5}$

   Proposed by PCChess

   Answer (B): The surface area is $2 \cdot 2^2\pi + 2\pi 2 \cdot 10 = 48\pi$. This means that each slice has 6π of frosting. Solving $6\pi = 2^2\pi + 2\pi 2h$, we get that $h = \boxed{\textbf{(B) } \frac{1}{2}}$.

10. For a positive composite integer n, let S be the set of divisors of n greater than 1 and less than n. Given that a and b are the smallest and largest elements of S, respectively, what is the sum of all n with $\frac{b}{a} = 15$?

    (A) 150 **(B)** 165 **(C)** 180 **(D)** 195 **(E)** 210

    Proposed by Emathmaster

    Answer (D): Suppose n has k divisors. Label the divisors as follows $1 = d_1 < d_2 < d_3 < \ldots < d_{k-1} < d_k = n$

    Clearly, $d_2 = a$ and $d_{k-1} = b$. In addition, $d_2 \cdot d_{k-1} = n$. In conjunction with $d_{k-1} = 15d_2$, this tells us that $n = 15 \cdot (d_2)^2$. Because n and d_2 are integers, n must have a factor of 3 and a factor of 5. This means we must have $d_2 = 2$ or $d_2 = 3$. These give us $n = 60$ and $n = 135$, respectively, so our answer is $\boxed{\textbf{(D) } 195}$.

11. Every element in nonempty set S is a distinct nonnegative integer less than or equal to 16. The product of the elements is not divisible by 8 and there are at most 2 odd numbers in S. Let N be the number of possible sets that can be S. Find the sum of the digits of N.

    (A) 12 **(B)** 13 **(C)** 15 **(D)** 16 **(E)** 17

    Proposed by kevinmathz

    Answer (A): Since S is not divisible by 8, the multiples of 8 are out of consideration. Now we check evens and multiples of 4. At the end, we multiply by $\binom{8}{0} + \binom{8}{1} + \binom{8}{2} = 37$.

Evens not divisible by 4: We have $2, 6, 10, 14$. Multiples of 4 but not 8: We have $4, 12$.

Our number can be odd, or even but not divisible by 8. Thus, we add our number of ways to choose this, which is thus $\binom{4}{2} + \binom{4}{1} + \binom{2}{1} + 1 = 13$. Our answer is thus $37 \cdot 13 - 1 = 480$. We subtract by 1 because we cannot include the empty subset. The sum digits of sum of digits is $\boxed{\textbf{(A) } 12}$.

12. Call two integers (a, b) *friends* if there is at least one integer x such that $(x-a)(x-b)$ is an integral power of 2. How many ordered pairs of *friends* (a, b) satisfy $1 \le a, b \le 7$?

    (A) 41 **(B)** 43 **(C)** 45 **(D)** 47 **(E)** 49

    Proposed by Emathmaster

    Answer (C): We see that the positive difference between a and b must be a difference of powers of 2, both less than 4 because our bound is 6. That leaves us the following differences: $0, 1, 2, 3, 4, 6$.

    We thus see there are a total of $14 - 2n$ ways to form the numbers if the differences is n so we sum up: $7 + 12 + 10 + 8 + 6 + 2 = \boxed{\textbf{(C) } 45}$.

13. Alice and Bob play a game. Alice goes first and they alternate between turns. In this game, an unfair coin is flipped. Alice wins if it is her turn and she flips heads; Bob wins if it is his turn and he flips tails. If the game is a fair game (i.e. both players have an equal chance of winning), what is the probability that the coin flips heads on a given flip?

    (A) $\frac{\sqrt{5}-1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{3-\sqrt{5}}{2}$ **(D)** $\frac{2}{5}$ **(E)** $\frac{1}{2}$

    Proposed by kevinmathz

    Answer (C): Let p be the probability that the coin flips heads. The probability Alice wins can be written as $p + pr + pr^2 + pr^3 + \cdots$, where $r = \text{ratio} = p(1-p)$ because Tail-Heads has to be rolled to cycle. With the formula, we see that we have the total probability as $\frac{p}{1-p(1-p)} = \frac{1}{2}$, so thus, $p^2 - 3p + 1 = 0$, so using the quadratic formula gets $p = \frac{3 \pm \sqrt{5}}{2}$. Since our answer is less than 1, it is $\boxed{\textbf{(C) } \dfrac{3-\sqrt{5}}{2}}$.

14. Let α and β be angles in Quadrants I or IV of the unit circle satisfying

    $$\log_3(\cos\alpha) + \log_3(\cos\beta) = -1 \text{ and } \cos(\alpha+\beta) = \frac{2}{15}.$$

    What is

    $$\frac{\tan(\alpha+\beta)}{\tan\alpha + \tan\beta}?$$

    (A) $\frac{3}{2}$ **(B)** $\frac{5}{3}$ **(C)** $\frac{5}{2}$ **(D)** 3 **(E)** 5

    Proposed by Emathmaster

Answer (C): Using logarithm rules for $\log_3(\cos\alpha)+\log_3(\cos\beta) = -1$, we get $\cos\alpha\cdot\cos\beta = \frac{1}{3}$. We can also expand $\cos(\alpha+\beta) = \frac{2}{15}$ as such: $\cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta = \frac{2}{15}$. Therefore, $\sin\alpha\cdot\sin\beta = \frac{1}{5}$.

Since $\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\cdot\tan\beta}$, we want the value of $\frac{1}{1-\tan\alpha\cdot\tan\beta}$.

Dividing the equation $\sin\alpha\cdot\sin\beta = \frac{1}{5}$ by $\cos\alpha\cdot\cos\beta = \frac{1}{3}$, we get $\tan\alpha\cdot\tan\beta = \frac{3}{5}$. Plugging this into $\frac{1}{1-\tan\alpha\cdot\tan\beta}$, we get $\boxed{\textbf{(C)}\ \frac{5}{2}}$.

15. Let τ be a function such that for all positive integers n, $\tau(n)$ denotes the number of positive divisors n has. Given that there are two possible values of n such that $\tau(n+1)-\tau(n) \geq 14$, where $n < 200$, what is the sum of the digits of the smaller value of n?

(A) 10 **(B)** 11 **(C)** 14 **(D)** 16 **(E)** 17

Proposed by Emathmaster

Answer (C): Clearly, $n = 1$ and $n = 2$ does not satisfy the inequality. Therefore, we will assume that $\tau(n) \geq 2$. This means that $\tau(n+1) \geq 16$. We will first search for integers with $\tau(n+1) = 16$. Note that $n+1$ is even if we elect for $\tau(n+1) = 16$ because that implies $\tau(n) = 2$, which means n is prime, and since 2 doesn't work, it must be an odd prime.

Now, we search for integers whose prime factorizations are in the form p^{15}, $p^7\cdot q$, $p^3\cdot q^3$, $p^3\cdot q\cdot r$, and $p\cdot q\cdot r\cdot s$. We'll focus more on the last 2, since they will most likely help us minimize n. We realize that $2\cdot 3\cdot 5\cdot 7 = 210$ is too big. We also realize that $2^3\cdot 3\cdot 5 - 1 = 119$ is also not prime. However, $2^3\cdot 3\cdot 7 - 1 = 167$ is prime.

We claim that the smallest possible value of n is 167.

Now, we will look at the other prime factorizations that give us $\tau(n+1) = 16$. Clearly, 2^{15} is much bigger than 200. $2^7\cdot 3 = 384$ is also too big. $2^3\cdot 3^3 = 216$ is also too big. If we try $n+1 = p^3\cdot q\cdot r$ with $q < r$, first we will assume $p = 2$ then assume $q = 2$. If $p = 2$ and $q > 3$, then $2^3\cdot 5\cdot 7 = 280 > 200$, so $p = 2$ implies $q = 3$. Since we already tested $2^3\cdot 3\cdot 5$ and $2^3\cdot 3\cdot 7$, we test $q = 11$. However, $2^3\cdot 3\cdot 11 = 8\cdot 33 > 200$ is too big. If we assume $q = 2$, then $n+1$ has to be at least $3^3\cdot 2\cdot 5 = 270 > 200$, which is a contradiction. Therefore, there are no other solutions with $\tau(n+1) = 16$ with $n < 200$.

If we have $\tau(n+1) = 17$, then $n+1 = p^{16}$ at the minimum. However, 2^{16} is much bigger than 200, so we suspect for a solution in $\tau(n+1) = 18$.

The minimum possible value of $n+1$ with $\tau(n+1) = 18$ must be $n+1 = p^2\cdot q^2\cdot r = 2^2\cdot 3^2\cdot 5 = 180$. Because $n = 179$, which is prime, $n = 179$ must be another solution. Since we are given that there are two solutions, $n = 167$ and $n = 179$ are the two solutions in question. The smaller is 167, which has a digit sum of $\boxed{\textbf{(C)}\ 14}$.

16. Big Zhao and Little Zhao are playing a game where they take turns tiling a n by n plane with circular tiles of radius $\frac{n}{10}$ where $n \geq 20$. No tiles can overlap or go off the edge. A player wins in this game if the other player is unable to place a tile during their turn. If Big Zhao starts first, and both players play using optimal strategy, who will win?

(A) Little Zhao will always win. **(B)** Big Zhao will always win.
(C) Little Zhao will win if and only if $\left\lceil \frac{n}{\pi} \right\rceil$ is even.

(D) Big Zhao will win if and only if $\lceil \frac{n}{\pi} \rceil$ is even.
(E) Little Zhao will win if and only if $n \leq 100$.

Proposed by jeteagle

Answer (B): We will prove Big Zhao will always win. First, let Big Zhao tile the center of this n by n plane. Now, every time Little Zhao places his tile, Big Zhao can tile his tile directly opposite of it from the center of the plane. If Little Zhao is able to tile his tile at some place, then Big Zhao will also because the tiling is symmetric across the center. Therefore, Big Zhao will be able to mirror Little Zhao's placements, and he will never run out of tiling places unless Little Zhao runs out first. This means $\boxed{\textbf{(B)}\text{ Big Zhao will always win.}}$

17. There are M polynomials $P(x)$ such that, for all real values of x,

$$(x^3 + x^2 - 4x - 4) \cdot P(x) = (x - 4) \cdot P(x^2),$$

and the leading coefficient of $P(x)$ is an integer with an absolute value of at most 5. Suppose the sum of all possible values of $P(3)$ is N. What is $M + N$?

(A) 1 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 264

Proposed by kevinmathz and Awesome_guy

Answer (D): We can factor $x^3 + x^2 - 4x - 4$ as follows:

$$x^2(x + 1) - 4(x + 1)$$

$$(x^2 - 4)(x + 1)$$

$$(x + 2)(x - 2)(x + 1)$$

Therefore, $(x + 2)(x - 2)(x + 1) \cdot P(x) = (x - 4) \cdot P(x^2)$.
Plugging in $x = \pm 2$, we get $0 = P(4)$. Therefore, $P(x) = Q(x)(x - 4)$ for some polynomial $Q(x)$. Substituting this back into our equation, we get:

$$(x + 2)(x - 2)(x + 1)(x - 4) \cdot Q(x) = (x - 4)(x^2 - 4) \cdot Q(x^2)$$

Cancelling the common factors on both sides, we get:

$$(x + 1) \cdot Q(x) = Q(x^2)$$

Plugging in $x = -1$ tells us that $0 = Q(1)$. Therefore, $Q(x) = (x-1)R(x)$ for some polynomial $R(x)$. Substituting this back and cancelling common factors, we get:

$$R(x) = R(x^2)$$

Since this relation must hold for all real x and because the degree of $R(x^2)$ is double the degree of $R(x)$, we must have $R(x)$ be a constant polynomial. Let $R(x) = a$ for some real number a. Retracing our steps:

$$P(x) = a(x - 4)(x - 1)$$

The leading coefficient in $P(x)$ is a, so a can be any integer from -5 to 5, inclusive. This counts the zero polynomial as well. Therefore, $M = 11$.

We also have that $P(3) = -2a$, but the sum of the integers from -5 to 5, inclusive is 0. Therefore, $N = 0$, so $M + N = \boxed{\textbf{(D) } 11}$.

18. Denote point C on circle ω with diameter $\overline{AB}$. The tangent lines to ω from A and C intersect at point D, with $BC = 5$ and $CD = AD = 3$. What is the length of $\overline{AB}$?

(A) 6 **(B)** $3\sqrt{5}$ **(C)** $5\sqrt{2}$ **(D)** $6\sqrt{2}$ **(E)** $9\sqrt{3}$

Proposed by Awesome_guy

Answer (B): Denote r as the radius of the circle. We seek the value of $2r$. Define O as the center of the circle. Denote E as the foot of the altitude from O to BC. E is the midpoint of BC.

Clearly, DO bisects $\angle ADC$. Let $\angle ADO = \angle ODC = \theta$. Because $\angle DAO = \angle DCO = 90$, we have that $\angle AOD = \angle DOC = 90 - \theta$. Because $\angle AOC + \angle COB = 180$ and OE bisects $\angle COB$, we have that $\angle COE = \theta$. Because $\angle DAO = \angle OEC = 90$, it follows that $\triangle DAO \simeq \triangle OEC$. We then proceed to length chase.

We know that $CE = \frac{5}{2}$ and $CO = r$. We also know that $DA = 3$ and $AO = r$. Through Pythagorean Theorem on $\triangle DAO$, we have that $DO = \sqrt{r^2 + 9}$. Using the similar triangles, $\frac{DO}{OA} = \frac{OC}{CE}$. Plugging in the known lengths and clearing denominators, we arrive at $2r^2 = 5\sqrt{r^2 + 9}$. After squaring both sides and moving all the terms to one side, we get $4r^4 - 25r^2 - 225 = 0$. Solving the quadratic in terms of r^2, we find that $r^2 = \frac{45}{4}$ (disregarding the negative root). It then follows that $r = \frac{3\sqrt{5}}{2}$ and our answer is $\boxed{\textbf{(B) } 3\sqrt{5}}$.

19. Let

$$(1 + 23i)(2 + 22i) \cdots (23 + i) = a + bi$$

for integers a and b. What is the remainder when $a - b$ is divided by 7?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Proposed by Emathmaster

Answer (A): Notice that for $n = 1, 2, 3, 4, ..., 11$:

$$(n + (24 - n)i)(24 - n + ni) = i((24 - n)^2 + n^2)$$

Therefore, $(1 + 23i)(2 + 22i) \cdots (23 + i) = (12 + 12i) \cdot \prod_{n=1}^{11} [i((24 - n)^2 + n^2)]$

Pulling the i out of the product:

$$= (12 + 12i) \cdot i^{11} \cdot \prod_{n=1}^{11} ((24 - n)^2 + n^2)$$

$$= (12 + 12i) \cdot -i \cdot \prod_{n=1}^{11} ((24 - n)^2 + n^2)$$

$$= (12 - 12i) \cdot \prod_{n=1}^{11}((24-n)^2 + n^2)$$

Let $P = \prod_{n=1}^{11}((24-n)^2 + n^2)$.

Then, $a = 12P$ and $b = -12P$, so we seek the remainder when $24P$ or $3P$ is divided by 7.

By bashing out P in mod 7, we arrive at $P \equiv 5 \pmod 7$, so $3P \equiv 1 \pmod 7$. Therefore, the remainder when $a - b$ is divided by 7 is $\boxed{\textbf{(A)}\ 1}$.

20. In convex quadrilateral $ABCD$, $\angle A = 90°$, $\angle C = 60°$, $\angle ABD = 25°$, and $\angle BDC = 5°$. Given that $AB = 4\sqrt{3}$, find the area of quadrilateral $ABCD$.

(A) 4 **(B)** $4\sqrt{3}$ **(C)** 8 **(D)** $8\sqrt{3}$ **(E)** $16\sqrt{3}$

Proposed by DeToasty3

Answer (D): We claim that if we reflect point C across the perpendicular bisector of line segment $\overline{BD}$ to get point C', then we get a right triangle ABC', where point D is on side AC'. We see that this happens because $\angle ABC' = \angle BDC' + \angle ABD = 5° + 25° = 30°$, $\angle BC'D = \angle BC'A = 60°$, and $\angle BAC' = 90°$. We also know that $\angle ADC' = \angle BDA + \angle BDC' = (180° - 90° - 25°) + (180° - 60° - 5°) = 65° + 115° = 180°$, so point D is on side AC'. By extension, we now know that right triangle ABC' is a $30 - 60 - 90$ right triangle, where $\angle A = 90°$, $\angle B = 30°$, and $\angle C' = 60°$.

We know that right triangle ABC' has the same area as quadrilateral $ABCD$ because triangles BCD and $BC'D$ have the same areas (this reflection preserves areas), and triangle ABD is unchanged. Since we are given that $AB = 4\sqrt{3}$, it follows that the other leg, AC', has length 4. We have that the area of right triangle ABC', and thereby the area of quadrilateral $ABCD$, is $\frac{1}{2} \cdot 4 \cdot 4\sqrt{3} = \boxed{\textbf{(D)}\ 8\sqrt{3}}$.

21. Consider a triangle $\triangle ABC$ with circumcircle ω, $AB = 13$, $AC = 5$, $BC = 12$. Let l be the line parallel to AB passing through C and let $l \cap \omega$ be P. Let the projection of P onto AC be D, and define E similarly for AB. Let PE meet ω again at K and let $PD \cap CK = G$. Then $GK = \frac{m}{n}$ where m and n are integers. Find $m + n$.

(A) 301 **(B)** 303 **(C)** 305 **(D)** 479 **(E)** 502

Proposed by realquarterb

Answer (A): Notice that CK is a diameter since $\angle CPK = 90°$ (by construction) and $CPKA$ is cyclic. Thus, $CK = 13$. Let F be the foot of the projection of P onto BC. Since $CDPF$ is a rectangle, $CD = PF$. Since $CPBA$ is an isosceles rectangle, we get that its height is $\frac{5}{13} * 12$ and $CP = \frac{119}{13}$ giving $PF = CD = \frac{595}{169}$. Then, by $\triangle CGD \sim \triangle CAK$, we get $CG = \frac{119}{13}$. Thus, $GK = GC + CK = \frac{119}{13} + 13 = \frac{288}{13}$. So the answer is $\boxed{\textbf{(A)}\ 301}$.

22. For positive integers m and n, define $f(m,n) = \left\lfloor \left(m+\frac{1}{n}\right)\left(n+\frac{1}{m}\right) \right\rfloor$. Then, let

$$S = \sum_{\substack{m,n>0 \\ m+n\le 2020}} f(m,n).$$

Find the sum of the digits of the remainder when S is divided by 1000. (Here, $\lfloor x \rfloor$ is the greatest positive integer less than or equal to x.)

(A) 8 **(B)** 10 **(C)** 12 **(D)** 14 **(E)** 16

Proposed by Emathmaster

Answer (E): We notice that by expanding, $f(m,n) = \left\lfloor \left(m+\frac{1}{n}\right)\left(n+\frac{1}{m}\right) \right\rfloor = mn+2+\lfloor \frac{1}{mn} \rfloor$. The value $\lfloor \frac{1}{mn} \rfloor$ evaluates to 0 if $mn > 1$. The only pair in our sum with $mn \le 1$ is $(1,1)$, in which the floor evaluates to 1. Therefore, $S = \sum_{\substack{m,n>0 \\ m+n\le 2020}} f(m,n) = 1 + \sum_{\substack{m,n>0 \\ m+n\le 2020}} mn+2$.

We will break up S into two parts:

$$A = \sum_{\substack{m,n>0 \\ m+n\le 2020}} mn$$

$$B = \sum_{\substack{m,n>0 \\ m+n\le 2020}} 2$$

To evaluate A, we first fix m and then plug in all the valid values of n:

$A = 1(1+2+3+4+...+2019)+2(1+2+3+4+...+2018)+3(1+2+3+4+...+2017)+...+2018(1+2)+2019(1$

Recall that $1+2+3+4+...+n = \frac{n(n+1)}{2} = \binom{n+1}{2}$.

Therefore,

$$A = 1\binom{2020}{2} + 2\binom{2019}{2} + 3\binom{2018}{2} + ... + 2018\binom{3}{2} + 2019\binom{2}{2}.$$

We can break up A as:

$$\binom{2020}{2} + \binom{2019}{2} + \binom{2018}{2} + ... + \binom{3}{2} + \binom{2}{2} = \binom{2021}{3}$$

$$+\binom{2019}{2} + \binom{2018}{2} + \binom{2017}{2} + ... + \binom{3}{2} + \binom{2}{2} = \binom{2020}{3}$$

$$+\binom{2018}{2} + \binom{2017}{2} + \binom{2016}{2} + ... + \binom{3}{2} + \binom{2}{2} = \binom{2019}{3}$$

$$+...$$

$$+\binom{3}{2}+\binom{2}{2}=\binom{4}{3}$$

$$+\binom{2}{2}=\binom{3}{3},$$

where the binomial coefficient on the right hand side of each equation follows from the Hockey Stick Identity. Then, we use the Hockey Stick Identity again:

$$A=\binom{2021}{3}+\binom{2020}{3}+\binom{2019}{3}+\ldots+\binom{4}{3}+\binom{3}{3}=\binom{2022}{4}$$

Now, we will evaluate B. To turn m and n from positive integers to nonnegative integers, make the substitution $m'=m-1$ and $n'=n-1$. Then, $m'+n'\le 2018$. Suppose that $2018-m'-n'=p$. Because $m'+n'\le 2018$, p is a nonnegative integer. Therefore, we have $m'+n'+p=2018$ for nonnegative integers m', n', and p. By stars and bars, we have $\binom{2020}{2}$. Since we add 2 for each valid (m,n) pair, we have that $B=2\binom{2020}{2}$.

Finally, $S\equiv 1+A+B \pmod{1000}$

$$S\equiv 1+\binom{2022}{4}+2\binom{2020}{2} \pmod{1000}$$

$$S\equiv 1+815+380\equiv 196 \pmod{1000}$$

The sum of the digits of 196 is $\boxed{\textbf{(E) } 16}$.

23. In triangle $\triangle ABC$, suppose $AB=2017$ and $AC=2020$. If I denotes the incenter of $\triangle ABC$, extend AI past I to intersect the circumcircle of $\triangle ABC$ again at D. If the area of $\triangle BIC$ is half of the area of $\triangle BCD$, $BC=\frac{m}{n}$ for relatively prime positive integers m and n. What is the remainder when $m+n$ is divided by 100?

(A) 77 **(B)** 78 **(C)** 79 **(D)** 80 **(E)** 81

Proposed by P_Groudon

Answer (A): Define E as the point of tangency with the incircle of $\triangle ABC$ and AC. Define F as the point of tangency with the incircle of $\triangle ABC$ and AB. Let G be the foot of the altitude from D to BC. Because $BD=CD$, G is the midpoint of BC.

Now, we will use the condition where the area of $\triangle BIC$ is half of the area of $\triangle BCD$. Denote r as the inradius of $\triangle ABC$. Since $\triangle BIC$ and $\triangle BCD$ share the same base and have an area ratio of $1:2$, the height of $\triangle BIC$ with respect to base BC is r. By the area ratio, $DG=2r$. By cyclic quadrilateral $ABCD$ and isosceles triangle $\triangle BCD$, $\angle GCD=\angle BCD=\angle DAE=\angle IAE$. Because $\angle IEA=\angle CGD=90^\circ$, $\triangle AEI\simeq\triangle CGD$. Suppose $AE=k$. By the similarity, $GC=2k$.

Now, we chase lengths. With $GC=2k$, we have $BC=4k$. By Two-Tangent Theorem, $4k=BC=BF+EC$. Clearly $BF=2017-k$ and $EC=2020-k$. Solving for $4k$, we get $\frac{8074}{3}$, so the answer is $\boxed{\textbf{(A) } 77}$.

24. Define $s(k)$ as the period of the decimal expansion of $\frac{1}{k}$. Let S be the set of integers that are greater than 1 and can be written in the form $3^a \cdot 7^b$, where a and b are nonnegative integers. What is the value of $\frac{1}{s(k)}$ summed over all $k \in S$?

(A) $\frac{19}{6}$ **(B)** $\frac{229}{72}$ **(C)** $\frac{27}{8}$ **(D)** $\frac{11}{3}$ **(E)** $\frac{271}{72}$

Proposed by AIME12345

Answer (C): We will break the sum $\displaystyle\sum_{k\in S} \frac{1}{s(k)}$ into three pieces:

$$\mathcal{A} = \sum_{a=1}^{\infty} \frac{1}{s(3^a)}$$

$$\mathcal{B} = \sum_{b=1}^{\infty} \frac{1}{s(7^b)}$$

$$\mathcal{C} = \sum_{a=1}^{\infty}\sum_{b=1}^{\infty} \frac{1}{s(3^a \cdot 7^b)}$$

For a certain decimal, let d be the integer formed by the repeating block of the decimal. For example, in the decimal 0.121212...., $d = 12$. Then, we can write $\frac{1}{k}$ as such:

$$\frac{d}{10^{s(k)}} + \frac{d}{10^{2s(k)}} + \frac{d}{10^{3s(k)}} + \cdots = \frac{1}{k}.$$

The LHS is an infinite geometric series with a first term of $\frac{d}{10^{s(k)}}$ and a common ratio of $\frac{1}{10^{s(k)}}$. By the formula for the sum of an infinite geometric series, we get:

$$\frac{d}{10^{s(k)} - 1} = \frac{1}{k}.$$

We clear the denominators to get:

$$10^{s(k)} - 1 = dk.$$

Because d is an integer, we can take both sides of the equation in modulo k:

$$10^{s(k)} - 1 \equiv 0 \pmod{k}.$$

Clearly, $s(k)$ equals the smallest positive integer n for which $10^n - 1 \equiv 0 \pmod{k}$ holds.

First, we will consider the evaluation of $s(3^a)$.

Since 3^1 and 3^2 divide $10 - 1 = 9$ but not 3^3, $s(3^1) = 1$ and $s(3^2) = 1$.

Define $v_p(n)$ as the largest nonnegative integer m for which p^m divides n. Since 3 is divisible by $(10 - 1)$, we may apply Lifting the Exponent.

$$v_3(10^{s(k)} - 1) = v_3(9) + v_3(s(k)).$$

Since $k = 3^a$ and we are working in modulo 3^a, we want $v_3(10^{s(k)} - 1) = a$.

Therefore, using $v_3(10^{s(k)} - 1) = v_3(9) + v_3(s(k))$, we find that $a - 2 = v_3(s(k))$.

The smallest possible value $s(k)$ can equal here is 3^{a-2}. Therefore, for $a \geq 3$, $s(3^a) = 3^{a-2}$, but $s(3) = 1$ and $s(9) = 1$.

It immediately follows that:

$$\mathcal{A} = \sum_{a=1}^{\infty} \frac{1}{s(3^a)} = 1 + (1 + \frac{1}{3} + \frac{1}{9} + ...) = \frac{5}{2}.$$

Next, we will use a similar process to evaluate $s(7^b)$.

We can't apply Lifting the Exponent directly because $(10-1)$ is not divisible by 7. However, a quick check shows us that $n = 6$ is the smallest positive integer n for which $10^n - 1$ is divisible by 7. Let $t(7^b) = \frac{s(7^b)}{6}$.

$$10^{6t(7^b)} - 1 \equiv 0 \pmod{7^b}.$$

Now, we may apply Lifting the Exponent:

$$v_7(10^{6t(7^b)} - 1) = v_7(999999) + v_7(t(7^b)).$$

We want $v_7(10^{6t(7^b)} - 1) = b$ because we are working in modulo 7^b. In addition $v_7(999999) = 1$. Therefore,

$$b - 1 = v_7(t(7^b))$$
$$t(7^b) = 7^{b-1}$$
$$s(7^b) = 6 \cdot 7^{b-1}.$$

Therefore, $\mathcal{B} = \sum_{b=1}^{\infty} \frac{1}{s(7^b)} = \frac{1}{6}(1 + \frac{1}{7} + \frac{1}{49} + ...) = \frac{7}{36}$

Now, we will evaluate $s(3^a \cdot 7^b)$. Clearly $s(3^a \cdot 7^b) = \text{lcm}(s(3^a), s(7^b)) = \text{lcm}(s(3^a), 6 \cdot 7^{b-1})$.

We remember earlier that $s(3^1) = s(3^2) = 1$. However, for $a \geq 3$, $s(3^a) = 3^{a-2}$. When $a = 1$ or $a = 2$, $v_p(s(3^a))$ is always strictly less than $v_p(6 \cdot 7^{b-1})$. However, when $a \geq 3$, we have $v_p(s(3^a)) \geq v_p(6 \cdot 7^{b-1})$.

Therefore, for $a = 1$ and $a = 2$: $\sum_{b=1}^{\infty} \frac{1}{s(3^a \cdot 7^b)} = \frac{7}{36}$.

However, for $a \geq 3$: $\sum_{b=1}^{\infty} \frac{1}{s(3^a \cdot 7^b)} = \frac{7}{36 \cdot 3^{a-3}}$.

For our final sum:

$$\mathcal{A} + \mathcal{B} + \mathcal{C} = \frac{5}{2} + 3 \cdot \frac{7}{36} + \sum_{a=3}^{\infty} \sum_{b=1}^{\infty} \frac{1}{s(3^a \cdot 7^b)}$$
$$\frac{37}{12} + \sum_{a=3}^{\infty} \frac{7}{36 \cdot 3^{a-3}}$$
$$\frac{37}{12} + \frac{7}{24}$$
$$\boxed{\textbf{(C)}\ \frac{27}{8}}.$$

25. For a given permutation of $1, 2, 3, 4, 5, 6$, denote a_n as the nth element in the permutation. A non-empty subset S of $\{1, 2, 3, 4, 5, 6\}$ has property P if for every k in S, the value a_k (not necessarily distinct from k) is also in S. In addition, a subset S has property Q if it has property P and no proper subsets of S have property P. For how many permutations of $1, 2, 3, 4, 5, 6$ do there exist sets A and B that satisfy the following conditions?

    (a) A and B contain no elements in common.

    (b) A and B both have property Q.

    (c) The union of A and B is $\{1, 2, 3, 4, 5, 6\}$.

    (A) 260 **(B)** 274 **(C)** 295 **(D)** 312 **(E)** 336

    Proposed by P_Groudon

    Answer (B): Draw a graph with 6 nodes and label them with the integers 1 through 6. Because a permutation is a bijective mapping of $\{1, 2, 3, 4, 5, 6\}$ onto itself, each node must have one directed edge entering the node and one directed edge exiting the node. The directed edge will point from n to a_n. For example, $a_1 = 4$ can be represented as the node with label 1 having a directed edge pointing to the node of label 4. Note that the directed edge can point back to its original node such as when we have $a_1 = 1$.

    We can show that when we draw a graph of the permutation. It will be comprised solely of disjoint loops, possibly more than 1.

    Take an arbitrary node. It can either point to itself, in which case it forms a loop on its own. If it points to a second node, that second node must point back to the first node or point to some other unused node. If it points to a third node, the third node can point back to the first node or some other unused node. This fashion continues. We can end up including all 6 nodes in a single continuous loop or end the loop prematurely. If we end the loop prematurely, the remaining nodes must also form loops. Therefore, our graph will be solely made up of disjoint loops. Because the permutation is bijective, every node will be included in some loop.

    A loop of a length of 1 node would appear as $a_i = i$ for some integer i in the interval $[1, 6]$. A loop of a length of 2 nodes would appear as $a_i = j$ and $a_j = i$ where $i \neq j$. A loop of a length of 3 nodes would appear as $a_i = j$, $a_j = k$, and $a_k = i$ where $i \neq j \neq k$ and so on for loops of longer lengths.

    Clearly, for a subset A of $\{1, 2, 3, ..., 6\}$ to have property P, A must contain some subset of the set of complete loops in the graph. If A contains an incomplete loop, then we can consider one of the elements that is in A and is a member of the incomplete loop. Then, one of those elements in A must map to an element outside of A, so that mapped element will be outside of A. This is a contradiction based on our definition of property P. Therefore, for a subset A to have property P it must contain some number of complete loops.

    We claim that a subset A of $\{1, 2, 3, ..., 6\}$ satisfies property Q if it contains only 1 complete loop. For the sake of the contradiction, assume that the elements of A are in more than 1 loop. That means we have at least 2 loops in the graph of A. Call the loops L_1, L_2, and so on. The subset containing the elements involved in L_1 satisfies property P because every element in the loop points to some other element in the loop, which is a contradiction. Therefore, A satisfies property Q if it represents 1 complete loop in the graph of the permutation.

    Now, we can begin counting the permutations. By the condition that A and B are disjoint and that they both have property Q, A and B must each correspond to distinct loops in the graph. To satisfy the condition where the union of A and B is $\{1, 2, 3, 4, 5, 6\}$, we must have exactly 2 loops in the graph.

There are 3 different ways we can break up the loop sizes: 5+1, 4+2, or 3+3.

Suppose we have n members to be put in a loop. Arrange the n elements in a circle where the order of the mapping is determined clockwise. For example, if 1, 3, and 5 appear in the circle in that clockwise order, that represents $a_1 = 3, a_3 = 5$, and $a_5 = 1$. There are $n!$ ways to arrange the elements in a circle. However, rotations of the arrangement do not matter, so we must divide by n. Therefore, the number of possible loops determined by n distinct elements is $(n-1)!$.

Case 1: 5+1

We have $\binom{6}{1}$ ways to choose the lone element and 1 way to arrange it in a circle. Then we have 4! ways to arrange the remaining 5 elements in a circle. Therefore, $\binom{6}{1} \cdot 1 \cdot 4! = 144$.

Case 2: 4+2

We have $\binom{6}{2}$ ways to choose the two elements and 1! way to arrange it in a circle. Then we have 3! ways to arrange the remaining 4 elements in a circle. Therefore, $\binom{6}{2} \cdot 1! \cdot 3! = 90$.

Case 3: 3+3

We have $\frac{1}{2} \cdot \binom{6}{3}$ ways to split the six elements into two groups of 3. We must divide $\binom{6}{3}$ by 2 because the order we choose the groups of 3 does not matter. For example, choosing $\{1, 2, 3\}$ and leaving $\{4, 5, 6\}$ for the other group is the same as choosing $\{4, 5, 6\}$ and leaving $\{1, 2, 3\}$ for the other group. There 2! ways to arrange a given loop of 3. Therefore, $\frac{1}{2} \cdot \binom{6}{3} \cdot (2!)^2 = 40$.

$144 + 90 + 40 = \boxed{\textbf{(B) } 274}$.