2020 TMC 12B Problems and Solutions Document

Olympiad Test Spring Series

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1. In a bag of marbles, $\frac{3}{4}$ of the marbles are blue. If $\frac{1}{3}$ of the blue marbles are taken out of the bag, what fraction of the remaining marbles in the bag are blue?

   (A) $\frac{1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{2}{3}$ **(E)** $\frac{3}{4}$

   Proposed by Emathmaster

   Answer (D): We have that $\frac{3}{4} \cdot \frac{1}{3} = \frac{1}{4}$ of the marbles are removed overall. This means $\frac{3}{4}$ of all the original marbles remain after taking out some of the blue ones. We also have that $\frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$ of the original marbles will remain in the bag and are blue. Therefore, the fraction of the remaining marbles that are blue is $\frac{1}{2} \div \frac{3}{4} = \boxed{\textbf{(D)}\ \frac{2}{3}}$.

2. In equilateral triangle ABC with $AB = 1$, let M denote the midpoint of $\overline{BC}$ and N denote the midpoint of $\overline{AC}$. What is the area of AMN?

   (A) $\frac{\sqrt{3}}{32}$ **(B)** $\frac{\sqrt{3}}{16}$ **(C)** $\frac{3\sqrt{3}}{32}$ **(D)** $\frac{\sqrt{3}}{9}$ **(E)** $\frac{\sqrt{3}}{8}$

   Proposed by PCChess

   Answer (B): Since M is the midpoint of $\overline{BC}$, the area of $\triangle CAM$ is half of $\triangle ABC$. Since N is the midpoint of $\overline{AC}$, the area of $\triangle AMN$ is half of $\triangle CAM$. Therefore, the area of $\triangle AMN$ is a quarter of $\triangle ABC$. We know that the area of $\triangle ABC$ is $\frac{\sqrt{3}}{4}$, so our answer is $\boxed{\textbf{(B)}\ \frac{\sqrt{3}}{16}}$.

3. The sum of consecutive prime numbers p and q is a multiple of 20. What is the least possible value of the product $p \cdot q$?

   (A) 51 **(B)** 77 **(C)** 221 **(D)** 667 **(E)** 899

   Proposed by Awesome_guy

Answer (E): Since p and q are consecutive primes, we can minimize their product by trying the minimize their sum. If $p+q=20$, the closest p and q can be are 13 and 7. However, this means that p and q are not consecutive, since 11 is prime. If $p+q=40$, the closest p and q can be are 17 and 23. Since, 19 is prime, this is not possible. If $p+q=60$, we have the solution $p=29$ and $q=31$. Therefore, the least possible value of $p \cdot q$ is $29 \cdot 31 = \boxed{\textbf{(E) } 899}$.

4. Let f be a function such that $f(f(x)) = \frac{x}{2} - 2$ for all real numbers x. If $f(f(f(f(y)))) = 100$ for some integer y, what is the sum of the digits of y?

(A) 4 **(B)** 7 **(C)** 8 **(D)** 10 **(E)** 11

Proposed by Emathmaster

Answer (B): Define $g(x) = f(f(x))$ for all real numbers x. Then, we have $g(x) = \frac{x}{2} - 2$ and $g(g(y)) = 100$. Evaluating the function once, we have $g(\frac{y}{2} - 2) = 100$. Evaluating the funtion again, we have $\frac{\frac{y}{2}-2}{2} - 2 = 100$. Solving for y, we get $y = 412$, so our answer is $\boxed{\textbf{(B) } 7}$.

5. Shenlar has a sum S that is initially set equal to 0. For each integer n from 1 to 100 inclusive, Shenlar adds the value of $\frac{n}{2^n}$ to S if and only if 2^n is divisible by n. When S is written in binary (base-two), what is the sum of the digits of S after the point? (For example, if the number was 0.01101_2, the sum would be 3.)

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Proposed by Radio2

Answer (A): Notice that for 2^n to be divisible by n, the prime factorization of n must consist solely of twos. Therefore, the only possible values of n are $1, 2, 4, 8, 16, 32,$ or 64. For each value of n, 2^n is clearly greater than n, which means that the value added to S will be less than one. Also, each value of $\frac{n}{2^n}$ is unique except the cases where $n = 1$ and $n = 2$. In that case, those possible values sum to one. Since there are 7 powers of 2 that are added to S, with one overlap, the value of S will have 6 ones. However, one of the ones will be before the radix point, so the answer is $\boxed{\textbf{(A) } 5}$.

6. Let rectangle $ABCD$ be such that $AB = CD = 8$ and $BC = AD = 5$. Construct four squares such that the sides of the rectangle are the diagonals of the squares. What is the sum of the areas of the regions that are in only one of the four squares?

(A) 75.5 **(B)** 80 **(C)** 84.5 **(D)** 89 **(E)** 93.5

Proposed by jeteagle

Answer (B): We see that the overlapping area is a square. We let the length of the diagonal of the square be x. We have $x = 4+4-5 = 3$, so the area of the small square is 4.5. The area of the 4 squares not counting overlaps is $8^2 + 5^2 = 89$. But since this counts the overlapped square twice, and we want to find the area of the non-overlapping parts, we have the answer be $89 - 2 \cdot 4.5 = \boxed{\textbf{(B) } 80}$.

7. Given that x and y are positive real numbers, with x and y each less than $\frac{\pi}{2}$, that satisfy the equations $x+y=\frac{\pi}{2}$ and $\sin(x)+2\cos(y)=\frac{3\sqrt{3}}{2}$, what is $|x-y|$?

   (A) 0 **(B)** $\frac{\pi}{12}$ **(C)** $\frac{\pi}{6}$ **(D)** $\frac{\pi}{4}$ **(E)** $\frac{\pi}{3}$

   Proposed by DeToasty3

   Answer (C): Note that since $x+y=\frac{\pi}{2}$, we have that $\cos(y)=\cos\left(\frac{\pi}{2}-x\right)=\sin(x)$. Then, we have that $3\sin(x)=\frac{3\sqrt{3}}{2}\to\sin(x)=\frac{\sqrt{3}}{2}\to x=\frac{\pi}{3}$. Then, $y=\frac{\pi}{6}$, so $|x-y|=\boxed{\textbf{(C) } \frac{\pi}{6}}$.

8. Let real numbers a,b,c,d satisfy the system of equations

$$\log_2 a+\log_3 b+\log_4 c=13$$
$$\log_2 a+\log_3 b+\log_5 d=11$$
$$\log_2 a+\log_4 c+\log_5 d=15$$
$$\log_3 b+\log_4 c+\log_5 d=12.$$

   Find the value of $a+b+c+d$.

   (A) 1722 **(B)** 4190 **(C)** 4762 **(D)** 7246 **(E)** 7256

   Proposed by jeteagle

   Answer (C): Adding all the equations, we get

$$3(\log_2 a+\log_3 b+\log_4 c+\log_5 d)=51 \implies \log_2 a+\log_3 b+\log_4 c+\log_5 d=17.$$

   Now, subtracting each equation from this one, we obtain

$$\log_5 d=4 \implies d=5^4=625$$
$$\log_4 c=6 \implies c=4^6=4096$$
$$\log_3 b=2 \implies b=3^2=9$$
$$\log_2 a=5 \implies a=2^5=32.$$

   Adding, we get $a+b+c+d=\boxed{\textbf{(C) } 4762}$.

9. Let O be the circumcenter of $\triangle ABC$ with circumradius 6. The internal angle bisectors of $\angle ABO$ and $\angle ACO$ meet $\overline{AO}$ at points D and E, respectively. If $AB=9$ and $AC=10$, then $DE=\frac{m}{n}$ for relatively prime positive integers m and n. What is $m+n$?

   (A) 23 **(B)** 29 **(C)** 33 **(D)** 41 **(E)** 57

   Proposed by Emathmaster and Ish_Sahh

   Answer (A): By the Angle Bisector Theorem on $\triangle ABO$, $\frac{AD}{DO}=\frac{AB}{BO}=\frac{3}{2}$. Because $AO=6$, we have $DO=\frac{12}{5}$. By the Angle Bisector Theorem on $\triangle ACO$, $\frac{OE}{EA}=\frac{CO}{CA}=\frac{3}{5}$. Therefore, $AE=\frac{15}{4}$. We also know that $DE=|AO-OD-AE|$. Plugging in the numbers, we get $DE=\frac{3}{20}$. The answer is $\boxed{\textbf{(A) } 23}$.

10. Let (B, J) be an ordered pair of positive integers such that either $B + J = 60$ or $BJ = 60$. Bela is assigned the number B and Jenn is assigned the number J. Each of them only knows their own number and that either $B + J = 60$ or $BJ = 60$. Once they are assigned their numbers, Bela says, "I don't know your number." Then, Jenn replies, "I don't know your number." How many possible ordered pairs (B, J) exist, if Bela and Jenn always tell the truth and are infinitely intelligent?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4 or more

Proposed by kevinmathz

Answer (C): First, if Bela says she doesn't know Jenn's number, then her number must be a factor of 60 and not equal to 60. Now, Jenn knows that Bela's number must be a factor of 60 not equal to 60 so if she doesn't know Bela's number, then her number is also a factor of 60 and not equal to 60. Now here's the catch: Jenn knows Bela's number is a factor of 60 and not equal to 60 so as a result, her number must be 30 or she would know that it's definitely the product. With Jenn's number as 30, Bela's number is 2 or 30 so our answer is $\boxed{\textbf{(C) } 2}$.

11. Arnold and Betty play a game. They each randomly write an integer between 1 and 6, inclusive, on a sheet of paper. Then, a fair six-sided dice is rolled. A person wins if the number they wrote down is closer to the number rolled on the dice than the other person's number is. The game results in a draw if the two numbers that were written are the same distance from the number rolled. What is the probability that Arnold wins?

(A) $\frac{1}{18}$ **(B)** $\frac{2}{9}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{7}{18}$ **(E)** $\frac{1}{2}$

Proposed by PCChess

Answer (D): We can simply solve for the probability of a draw and the subtract from 1 and divide by 2.

Case 1 - They both write the same number: This obviously results in a draw. There is a $\frac{1}{6}$ chance that they write the same number.

Case 2 - The difference between the 2 numbers is 2: There are 4 possibilities. $(1, 3), (2, 4), (3, 5), (4, 6)$. Each time, the roll must be the number between the 2 numbers written down. Hence, for each possibility, the probability that it occurs is $\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{216}$. There are 8 possibilities, so we multiply by 8 to get $\frac{8}{216} = \frac{1}{27}$.

Case 3 - The difference between the 2 numbers is 4: There are 2 possibilities. $(1, 5), (2, 6)$. The roll must be in the middle of the 2 numbers. The probability is $4 \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{54}$.

Summing, the probability of a draw is $\frac{1}{6} + \frac{1}{27} + \frac{1}{54} = \frac{12}{54} = \frac{2}{9}$. The probability that Arnold wins is therefore $\boxed{\textbf{(D) } \dfrac{7}{18}}$.

12. For how many ordered pairs of positive integers (A, B), with $A \leq 3$ and $B \leq 2$, does the equation $x^3 - Ax - B = 0$ have a real solution x in the interval $[1.5, 2)$?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Proposed by Emathmaster

Answer (D): We see that if (A, B) is one of $(1, 2)$, $(2, 1)$, $(2, 2)$, or $(3, 1)$, then the Intermediate Value Theorem suffices. For $(3, 2)$, we see that $x = 1.5$ yields a negative value, and $x = 2$ yields 0, and for $(1, 1)$, we see that $x = 1.5$ and $x = 2$ both yield positive values. As we increase x from 1.5 to 2, we see that x^3 increases at a faster rate than $Ax + B$ since A is relatively small ($2^2 > 1.5^2 > 3 > 1$). Therefore, the difference between x^3 and $Ax + B$ will increase, so $x^3 - Ax - B$ will increase. Therefore, $x^3 - Ax - B$ is strictly increasing on the interval $[1.5, 2)$, with no real roots to be found. Therefore, our final answer is $\boxed{\textbf{(D)}\ 4}$.

13. Edwin has two chess pieces that he places both on the center square of a 5×5 chessboard. He sets a border one square wide on the edges of the chessboard, leaving a 3×3 area in the middle. In one move, each piece moves as follows:

- The white piece moves one square either vertically or horizontally and then two squares in a perpendicular direction.
- The black piece moves one square either vertically or horizontally.

Each piece moves repeatedly until it first lands on a square in the border, at which point it stops moving. If both pieces move randomly but always abide by their rules, what is the probability that the white and black pieces will end up on the same square after they each stop moving?

(A) $\frac{1}{64}$ **(B)** $\frac{1}{16}$ **(C)** $\frac{1}{9}$ **(D)** $\frac{1}{4}$ **(E)** $\frac{1}{2}$

Proposed by Radio2

Answer (B): We assign coordinates, with the center square at $(0, 0)$.

We see that the white piece immediately moves onto the border, onto one of the eight squares $(\pm 2, \pm 1)$ or $(\pm 1, \pm 2)$, with the probability of being on any one of these squares being $1/8$. We must thus compute the probability that the black piece ends on one of these squares.

Exploiting symmetry, we see that the probabilities of the black piece ending on: $(2, 0)$, $(-2, 0)$, $(0, 2)$, $(0, -2)$ are equal, as are the probabilities of ending on these eight squares: $(\pm 2, \pm 1)$ or $(\pm 1, \pm 2)$.

We see that no matter what, after the first move the black piece is adjacent to one of the four squares $(2, 0), (-2, 0), (0, 2), (0, -2)$. Now let the probability of ending on one of these four squares be P. We see that there is a $1/4$ chance the black piece moves directly onto one of these squares, a $1/4$ chance it moves back to the origin, and a $1/2$ chance it moves to $(\pm 1, \pm 1)$. From the origin, the piece's only move is back to a square adjacent to one of the four, so this case's probability is $P/4$. From $(\pm 1, \pm 1)$ there is a $1/2$ chance the piece goes onto a border square and a $1/2$ chance it returns to a square adjacent to one of the four. So this case's probability is $(1/2)(1/2)P = P/4$.

Thus $P = 1/4 + P/4 + P/4$, so $P = 1/2$. Thus the probability of ending on a square the white piece can also end on is $1 - 1/2 = 1/2$, and the probability both pieces land on the same square is thus $\boxed{\textbf{(B)}\ \dfrac{1}{16}}$.

14. When the solutions to $y^4 + y^2(-1 + 2i\sqrt{3}) - 2 + 2i\sqrt{3} = 0$ are plotted in the complex plane, the area of the polygon with the roots as its vertices is A. Find A^2.

(A) 12 **(B)** 16 **(C)** 21 **(D)** 24 **(E)** 36

Proposed by Ish_Sahh

Answer (A): Treat the polynomial as a quadratic in terms of y^2. We first notice that $y^2 = -1$ is a solution to the polynomial. This gives us $y = \pm i$. By Vieta's formulas, the other solution for y^2 is given by $(-1) \cdot y^2 = -2 + 2i\sqrt{3}$ or $y^2 = 2 - 2i\sqrt{3}$. This means that $y^2 = 4\text{cis}(\frac{4\pi}{3})$. When we take the square root of a complex number, we take the square root of its magnitude and take half of the angle. So $y = 2\text{cis}(\frac{2\pi}{3})$ or $y = 2\text{cis}(\frac{5\pi}{3})$. These complex numbers are both the same distance away from the imaginary axis, which contains our first two solutions. This common distance is the absolute value of the real part of y. They are also on opposite sides of the imaginary axis.

Therefore, the polygon can be broken into two triangles with the same base and height. The base is the distance between $-i$ and i, which is 2, while the absolute value of the real part of $2\text{cis}(\frac{2\pi}{3})$ is the height. Our area is given as $2|\text{Re}(2\text{cis}(\frac{2\pi}{3}))|$. We easily find that $|\text{Re}(2\text{cis}(\frac{2\pi}{3}))| = \sqrt{3}$, so $A = 2\sqrt{3}$ and $A^2 = 12$.

Our answer is $\boxed{\textbf{(A)}\ 12}$.

15. There are two 2-digit prime numbers p less than 30 such that when the quantity $2^{2020} + 19^{1010}$ is divided by p, the result is an integer. What is the sum of these primes?

(A) 30 **(B)** 36 **(C)** 42 **(D)** 46 **(E)** 52

Proposed by andyxpandy99

Answer (C): Observe that $2^{2020} + 19^{1010} = 16^{505} + 361^{505}$. Note that we can factor the RHS as

$$(16 + 361)(16^{504} - 16^{503} \cdot 361 + \cdots - 16 \cdot 361^{504} + 361^{505}).$$

$16 + 361 = 377 = 13 \cdot 29$. Now we know that 13 and 29 divide $2^{2020} + 19^{1010}$. We are given that there are two 2-digit primes less than 30 that divide $2^{2020} + 19^{1010}$, so our two primes must be 13 and 29. Our answer is $\boxed{\textbf{(C)}\ 42}$.

16. In triangle ABC with $AB = 2$ and $AC = 4$, construct squares $YZAB$, $WXAC$, and $UVBC$ on the exteriors of sides $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$, respectively. Given that the sum of the squares of the side lengths of hexagon $XWUVYZ$ is 128, the length of $\overline{BC}$ can be expressed as $a\sqrt{b}$, where a and b are positive integers and b is not divisible by the square of any prime. Compute the value of $a + b$.

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Proposed by Awesome_guy

Answer (A): Clearly, $YZ^2 = AB^2$, $XW^2 = AC^2$, and $VU^2 = BC^2$.

So we can write the equation $(AB^2 + AC^2 + BC^2) + (ZX^2 + WU^2 + VY^2) = 128$.

Now, we will find the remaining squares of the sides of the hexagon using Law of Cosines.

Suppose $\angle ABC = \theta$. Because $\angle ABC + \angle ABY + \angle CBV + \angle YBV = 360^\circ$ and $\angle ABY = \angle CBV = 90^\circ$, we have that $\angle YBV = 180^\circ - \theta$. We know that $-\cos(\theta) = \cos(180 - \theta)$, $YB = AB$, and $BV = BC$. Therefore, by Law of Cosines on $\triangle YBV$, we find that $VY^2 = AB^2 + BC^2 + 2 \cdot AB \cdot BC \cdot \cos\theta$. By Law of Cosines on $\triangle ABC$, $\cos\theta = \frac{AB^2+BC^2-AC^2}{2 \cdot AB \cdot BC}$. We ultimately obtain $VY^2 = 2(AB^2 + BC^2) - AC^2$.

Similarly, $WU^2 = 2(AC^2+BC^2) - AB^2$ and $ZX^2 = 2(AB^2+AC^2) - BC^2$. Substituting these back into our original equation, we obtain $4(AB^2 + AC^2 + BC^2) = 128$. Because $AB^2 = 16$ and $AC^2 = 4$, it follows that $BC^2 = 12$ or $BC = 2\sqrt{3}$. Therefore, our answer is $\boxed{\textbf{(A) } 5}$.

17. There are n integers x in the interval $2 \le x \le 2020$ such that when each of the values

$$\left\lfloor \frac{x^{10}}{x-1} \right\rfloor \text{ and } \left\lfloor \frac{x^9}{x-1} \right\rfloor$$

are divided by 48, their remainders are equal. What is the sum of the digits of n? (Here, $\lfloor r \rfloor$ denotes the greatest integer less than or equal to r.)

(A) 12 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16

Proposed by Ish_Sahh

Answer (A): This equation can be rewritten as $\lfloor \frac{x^{10}-1}{x-1} + \frac{1}{x-1} \rfloor \equiv \lfloor \frac{x^9-1}{x-1} + \frac{1}{x-1} \rfloor \pmod{48}$. Since $x-1$ is a factor of $x^{10}-1$ and x^9-1, both sides can be expanded to $\lfloor x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 + \frac{1}{x-1} \rfloor \equiv \lfloor x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 + \frac{1}{x-1} \rfloor \pmod{48}$. Each of the terms x^n for integral x, n are integers, so they can be brought outside of the floor function. This would make the equation $x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 + \lfloor \frac{1}{x-1} \rfloor \equiv x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 + \lfloor \frac{1}{x-1} \rfloor \pmod{48}$. After like terms are cancelled out, all that's left is $x^9 \equiv 0 \pmod{48}$, which says that x^9 is divisible by 48. Since $48 = 2^4 \cdot 3^1$, all that is needed is x to be divisible by 6. With $1 < x < 2020$, there are $\lfloor \frac{2020}{6} \rfloor - \lfloor \frac{1}{6} \rfloor = 336$ values for x that are divisible by 6 and satisfy the equation. The digit sum of 336 gives the answer of $\boxed{\textbf{(A) } 12}$.

18. A pyramid $ABCDE$ has square base $ABCD$ and apex E such that $AE = BE = CE = DE$. Suppose the planes determined by $\triangle ABE$ and $\triangle BCE$ form a 120° angle. If $AB = 2\sqrt{3}$, what is AE?

(A) $\sqrt{3}$ **(B)** 3 **(C)** $2\sqrt{3}$ **(D)** 4 **(E)** $2\sqrt{5}$

Proposed by P_Groudon

Answer (B): In the plane of face $\triangle ABE$, let the foot of the altitude from A to BE be F. By symmetry, the foot of the altitude from C to BE along the plane of face $\triangle BCE$ is also F. In addition $AF = FC$. Because the planes form a 120° angle, $\triangle AFC$ is an isosceles triangle with $\angle AFC = 120^\circ$. Because AC is the diagonal of square $ABCD$ with $AB = 2\sqrt{3}$, we have $AC = 2\sqrt{6}$. By the 120-30-30 triangle, $AF = 2\sqrt{2}$. By Pythagorean Theorem on $\triangle AFB$, we have $FB = 2$. Let $AE = x$. Then, $EF = x - 2$. By Pythagorean Theorem on $\triangle AEF$, we have $(x-2)^2 + 8 = x^2$, solving for x, we find $x = 3$, so our answer is $\boxed{\textbf{(B) } 3}$.

19. Let $\overline{BC}$ be a line segment, and let F be a point on $\overline{BC}$. Construct points A and D, and let E be the intersection of $\overline{AC}$ and $\overline{BD}$ so that A, D, and E are all on the same side of $\overline{BC}$, and $\overline{AB} \parallel \overline{CD} \parallel \overline{EF}$. Given that lengths AB and CD are positive integers and $EF = 24$, find the number of possible ordered pairs of lengths (AB, CD). (Assume that all six points are distinct.)

(A) 15 **(B)** 18 **(C)** 21 **(D)** 24 **(E)** 27

Proposed by Emathmaster

Answer (C): Notice that $\triangle ABC \simeq \triangle EFC$. Therefore, we may write $\frac{FC}{BC} = \frac{EF}{AB}$. In addition, $\triangle BCD \simeq \triangle BFE$. Therefore, we may write $\frac{BF}{BC} = \frac{EF}{DC}$. Adding our two equations together:

$$\frac{1}{BC}(FC + BF) = EF(\frac{1}{AB} + \frac{1}{DC})$$

Because $FC + BF = BC$, the left hand side becomes 1. (The result we have shown is more commonly known as the Crossed Ladders Theorem). Plugging in $EF = 24$ and clearing the denominators:

$$AB \cdot DC - 24 \cdot (AB + DC) = 0$$

We can use Simon's Favorite Factoring Trick to factor:

$$(AB - 24)(DC - 24) = 576$$

576 cannot be written as the product of two negative numbers, both of which have absolute values less than 24. Therefore, both factors must be positive. Now, we must count the number of positive integer divisors of $576 = 2^6 \cdot 3^2$. Finally, $(6+1)(2+1) = \boxed{\textbf{(C) } 21}$.

20. Richard writes the quadratic $f(x) = ax^2 + bx + c$ on a whiteboard, where a, b, and c are distinct nonzero complex numbers. Matthew sees Richard's quadratic, and rearranges the order of the coefficients (i.e. permutes the order of a, b, and c) to make his own six distinct quadratics: $g_1(x)$, $g_2(x)$, $g_3(x)$, $g_4(x)$, $g_5(x)$, and $g_6(x)$ (one of which is equal to $f(x)$). What is the minimum number of possible distinct roots of

$$\prod_{k=1}^{6}(f(x) + g_k(x))?$$

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 10

Proposed by DeToasty3 and P_Groudon

Answer (C): Because each of a, b, and c appear once as coefficients in each of $f(x)$ as well as the $g(x)$ quadratics, we can scale the coefficients such that $a = 1$. Scaling the coefficients also has no effect on the roots.

We notice that the coefficients don't change between $f(x)$ and the $g(x)$ polynomials. Therefore, if $f(1) = 0$ then $g_k(1) = 0$ for all integers $1 \leq k \leq 6$. Therefore, we can cut down on the roots by electing 1 as a root of $f(x)$.

Now, we rewrite $f(x) = x^2 - (c+1)x + c$. To prevent equal coefficients or zero coefficients, we must establish that $c \neq -1, \frac{-1}{2}, 0, 1, 2$. So the set of coefficients is $\{1, -c-1, c\}$. Since we know that 1 is a root for all of the $g(x)$ quadratics, we only need to compute the coefficient of x^2 and the constant term for each quadratic and then use Vieta's formulas to find the other root.

Adding each of $\{1, -c-1, c\}$ to 1, our set of possibilities for the coefficient of x^2 is $\{2, c+1, -c\}$. Adding each of $\{1, -c-1, c\}$ to c, our set of possibilities for the constant term is $\{c+1, -1, 2c\}$.

Now, we go through all the possibilities for the other root of the $g_k(x)$ coefficients by matching an element of $\{2, c+1, -c\}$ with an element of $\{c+1, 2c, -1\}$. Note that we can't match up 2 and $c+1$, $c+1$ and $2c$, and $-c$ and -1 because those pairs were generated from adding the same corresponding coefficient of $f(x)$.

By computing out the different possibilities for the roots;

$2, 2c \rightarrow c$

$2, -1 \rightarrow \frac{-1}{2}$

$c+1, c+1 \rightarrow 1$

$c+1, -1 \rightarrow \frac{-1}{c+1}$

$-c, c+1 \rightarrow \frac{c+1}{-c}$

$-c, 2c \rightarrow -2$

c can't equal any of 1, $\frac{-1}{2}$, or -2, so we have at least 4 distinct values. If we set $c = \frac{-1}{c+1}$, we get $c = \text{cis}(\frac{2\pi}{3}) = \frac{-1}{c+1} = \frac{c+1}{-c}$. We can also have $c = \text{cis}(\frac{4\pi}{3})$ and have the three expressions equal the same value.

We claim 4 is the minimum.

Proof: From our work above, $\{1, \frac{-1}{2}, -2\}$ are in the set of roots no matter what c is. Because c does not equal one of those three numbers, we must have a fourth distinct element. However, we assumed near the beginning that one of the roots to $f(x)$ had to be 1. However, what if $f(x)$ did not have 1 as a root?

Consider $f(x) = g_1(x) = ax^2 + bx + c$ and $g_2(x) = ax^2 + cx + b$. We claim that if 1 isn't a root of $f(x)$, $(f(x) + g_1(x))(f(x) + g_2(x))$ has exactly four distinct roots. Suppose that there was some root $r \neq 1$ that a root of both $f(x)$ and $g_2(x)$. Then, $ar^2 + br + c = ar^2 + cr + b$. This rearranges to $(b-c)(r-1) = 0$. The coefficients must be distinct, so this means that $r = 1$, which is a contradiction. Therefore, if $r \neq 1$, $f(x)$ and $g_2(x)$ have separate roots, which creates four distinct roots. However, this means that we cannot possibly do any better than 4 distinct complex roots. Therefore, the minimum is indeed $\boxed{\textbf{(C) } 4}$.

21. In a room with 10 people, each person knows exactly 4 different languages. A conversation is held between every pair of people with a language in common. If a total of 36 different languages are known throughout the room, and no two people have more than one language

in common, what is the sum of all possible values of n such that a total of n conversations are held?

(A) 19 **(B)** 25 **(C)** 32 **(D)** 35 **(E)** 37

Proposed by DeToasty3

Answer (C): We have that 10 people speak 4 languages. Label each of the $10 \cdot 4 = 40$ languages associated with a person with a token. Since there are 36 languages, each of the languages must have at least one token. After we give each language a token, we have $40 - 36 = 4$ remaining tokens left to distribute. Since no two people have more than one language in common, we may assume that for any language with k people speaking it, where $k \geq 2$, $\binom{k}{2}$ is the number of conversations held, with no need to subtract "extra" conversations for if two people had conversations in two or more different languages. If $k = 1$, then no conversations take place. Our goal is to find all of the cases that result after distributing 4 tokens into 36 languages, where the languages are indistinguishable, and see how many conversations happen in each case.

Case 1: One language gets all four tokens. Then, a total of $4 + 1 = 5$ tokens are in that language, giving us $\binom{5}{2} = 10$ conversations.

Case 2: One language gets three tokens, and one other language gets one token. Then, a total of $3 + 1 = 4$ tokens are in one language, and a total of $1 + 1 = 2$ tokens are in the other language, giving us $\binom{4}{2} + \binom{2}{2} = 6 + 1 = 7$ conversations.

Case 3: Two languages get two tokens each. Then, a total of $2 + 1 = 3$ tokens are in each language, giving us $2\binom{3}{2} = 6$ conversations.

Case 4: One language gets two tokens, and two other languages get one token each. Then, a total of $2 + 1 = 3$ tokens are in one language, and a total of $1 + 1 = 2$ tokens are in the other two languages, giving us $\binom{3}{2} + 2\binom{2}{2} = 3 + 2 = 5$ conversations.

Case 5: Four languages get one token each. Then, a total of $1 + 1 = 2$ tokens are in the four languages, giving us $4\binom{2}{2} = 4$ conversations.

Note that in each case, it is possible to assign each token to one of the ten people such that the problem's conditions are satisfied, so the sum of all possible values of n is $10 + 7 + 6 + 5 + 4 = \boxed{\textbf{(C) } 32}$.

22. Let $(a \oplus b)$ denote the bitwise exclusive-or (XOR) of a and b. This is equivalent to adding a and b in binary (base-two), but discarding the "carry" to the next place value if it is applicable. For instance, $(1_2 \oplus 1_2) = 0_2$, $(1_2 \oplus 0_2) = 1_2$, and $(5 \oplus 3) = (101_2 \oplus 011_2) = 110_2$. How many ordered pairs of nonnegative integers (x, y) both less than 32 satisfy $(x \oplus y) > x \geq y$?

(A) 63 **(B)** 99 **(C)** 127 **(D)** 155 **(E)** 255

Proposed by PCChess

Answer (D): First, we notice that when we apply $x \oplus y$ for binary numbers x and y, the addition in one place has no effect on the addition in other places because there are no carries. Clearly, x and y cannot have the same number of digits in binary. This is because if they did have the same number of digits (when leading zeroes are disregarded), both would have a leading 1 and then $x \oplus y$ would have a leading 0. This would cause $x \oplus y$ to be less than x, which is bad.

It follows that y must have fewer digits than x in binary. To make the following claim, we will disregard leading zeroes in x and y. We know y must have a leading 1. Suppose that leading 1 is in the ds place. If x has a 1 in the ds place. Then, $x \oplus y$ will have a 0 in the ds place. Because everything to the left of the ds place will be the same between x and $x \oplus y$, this causes $x > x \oplus y$.

However, if x has a 0 in the ds place, $x \oplus y$ will have a 1 in the ds place. Again, because everything to the left of the ds place will be the same between x and $x \oplus y$, this causes $x \oplus y > x$.

We will do casework based on the number of digits of y when y is written in binary. Now we will regard leading 0s only for x. Note that x can have up to 5 digits.

Case 1: y has 4 digits

That means x looks like:

_0___

On the other hand, y looks like:

1___

Note that the digit to the left of 0 in x must be a 1. We may assign whatever we want to the remaining 6 blank spaces between x and y because the addition in those places have no effect on the addition in the other places. We have $2^6 = 64$ cases here.

Case 2: y has 3 digits

That means x looks like:

__0__

On the other hand, y looks like:

1__

Consider the two blank spaces to the left of the 0 in x. We can opt for leading zeroes. For example, we can have 010__, which is the equivalent to 10__ while still satisfying the condition. However, the two blank spaces cannot be both zero. So we have $2^2 - 1$ ways to fill in those two blank spaces. Again, we can do whatever we want with the remaining 4 blank spaces. So we have $3 \cdot 2^4 = 48$ cases here.

Case 3: y has 2 digits

x : ___0_

y : 1_

Again, the three blank spaces to the left of the zero in x can't ALL be 0. So we have $2^3 - 1$ ways to fill in those spaces. We can do whatever we want with the other two spaces. So we have $7 \cdot 2^2 = 28$ cases here.

Case 4: y has 1 digit

x : ____0

y: 1

The four blank spaces in x cannot all be 0, so we have 15 cases here.

We have $\boxed{\textbf{(D) } 155}$ cases total.

23. A real number x in the interval $0 \le x \le \frac{\pi}{2}$ satisfies the equation

$$\sin(x + \pi \sin(x)) = \cos(x + \pi \cos(x)).$$

Then, the value of $\sin(x)$ can be written as $\frac{a+\sqrt{b}}{c}$, where a, b, and c are positive integers, a and c are relatively prime, and b is not divisible by the square of any prime. What is $a^2 + b^2 + c^2$?

(A) 26 **(B)** 42 **(C)** 52 **(D)** 66 **(E)** 86

Proposed by Ish_Sahh

Answer (D): Using the identity $\sin(x) = \cos(x - \frac{\pi}{2})$, the equation would be $\cos(x + \pi\cos(x) - \frac{\pi}{2}) = \cos(x + \pi\cos(x))$. If $\cos a = \cos b$, then either $a = b$ or $a = -b$ with $-\pi, a, b < \pi$.

Case $1(a = b)$: $x + \pi\cos(x) + \frac{\pi}{2} = x + \pi\sin(x)$: Cancelling out x on both sides and dividing by $-\pi$ would give $\cos x = \sin x + \frac{1}{2}$. Squaring both sides and using the identity $\cos^2 x = 1 - \sin^2 x$ gives $1 - \sin^2 x = \sin^2 x + \sin x + \frac{1}{4}$. This simplifies to $2\sin^2 x - \sin x - \frac{3}{4} = 0$. Using quadratics would give to sole positive answer of $\frac{1+\sqrt{7}}{4}$.

Case $2(a = -b)$: $x + \pi\cos(x) - \frac{\pi}{2} = -x - \pi\sin(x)$: This would impose $2x + \pi(\cos x + \sin x) = \frac{\pi}{2}$. However, if $0 \le x \le \frac{\pi}{2}$, then $\sin x + \cos x \ge 1$ and $2x \ge 0$. This wouldn't work as $\frac{\pi}{2} = 2x + \pi(\cos x + \sin x) \ge 0 + \pi(1) = \pi$ would have to have a solution. This isn't true, so this case has no solutions.

Our only solution is $\sin x = \frac{1+\sqrt{7}}{4}$ which would give $a = 1, b = 7, c = 4$. Therefore, the solution is $1^2 + 7^2 + 4^2 = \boxed{\textbf{(D) } 66}$.

24. Let O_1 be a circle with radius r. Let O_2 be a circle with radius between $\frac{r}{2}$ and r, exclusive, that goes through the center of circle O_1. Denote points X and Y as the intersections of the two circles. Let P be a point on the major arc $\widehat{XY}$ of O_1. Let $\overline{PX}$ intersect O_2 at A, strictly between P and X. Let $\overline{PY}$ intersect O_2 at B, strictly between P and Y. Let E be the midpoint of $\overline{PX}$ and F be the midpoint of $\overline{PY}$. If $AY = 100$, $AB = 65$, and $EF = 52$, what is BX?

(A) 104 **(B)** 105 **(C)** 106 **(D)** 107 **(E)** 108

Proposed by jeteagle, P_Groudon, and DeToasty3

Answer (B): Denote C_1 as the center of O_1. Because C_1AXY is cyclic and PC_1X is isosceles with $PC_1 = C_1X$, we have $\angle C_1YA = \angle C_1XA = \angle XPC_1 = \angle APC_1$. Using $\angle C_1YA = \angle APC_1$ and $PC_1 = C_1Y$, it follows that $PA = AY$, so AC_1 is perpendicular to PY. Because PY is a chord of the circle and AC_1 passes through the center of the circle, AC_1 must pass through F. Similarly, BC_1 must pass through E.

Therefore, we have that $\triangle PAB$ has orthocenter C_1. From $\triangle PBX$ having $PB = BX$ and $\triangle PAY$ having $PA = AY$, we are really after PB. We are given that $AY = PA = 100$. Clearly, $\triangle PEF \simeq \triangle PBA$, so $\frac{PF}{PA} = \frac{EF}{AB}$. Plugging in $PA = 100$, $AB = 65$, and $EF = 52$, this tells us that $PF = 80$. Through Pythagorean Theorem on $\triangle APF$, we get $AF = 60$. Then, by Pythagorean Theorem on $\triangle AFB$, we get $FB = 25$. Therefore, $PB = BX = \boxed{\textbf{(B) } 105}$.

25. There are N ordered pairs (x, y) of integers with $0 \le x, y < 39375$ such that when each of the values
$$(x+y)^3 \text{ and } (x^3+y^3)$$
are divided by 39375, their remainders are equal. How many positive integer divisors does N have?

(A) 72 **(B)** 96 **(C)** 108 **(D)** 144 **(E)** 192

Proposed by Emathmaster

Answer (D): The given condition is equivalent to $xy(x+y) \equiv 0 \pmod 3$, $xy(x+y) \equiv 0 \pmod 7$, and $xy(x+y) \equiv 0 \pmod{5^4}$.

We can count the number of possible residues (x, y) mod 3 with PIE, as at least one of $x, y, x+y$ is divisible by 3: If $x \equiv 0 \pmod 3$ then there are 3 possible values for y, if $y \equiv 0 \pmod 3$ then there are 3 possible values for x, if $x+y \equiv 0 \pmod 3$ then there are three possible pairs $(x, y) \pmod 3$. This gives a sum so far of $3+3+3=9$. When $x \equiv y \equiv 0 \pmod 3$, we get one pair (x, y), when $x+y \equiv x \equiv 0 \pmod 3$ as well as $x+y \equiv y \equiv 0 \pmod 3$ we get one more pair. Our PIE count so far is now $9-1-1-1=6$. Finally, when $x+y \equiv x \equiv y \equiv 0 \pmod 3$ we get an additional one, for a total of 7 pairs.

Similarly, the number of pairs $(x, y) \pmod 7$ is 19. We can now tackle the main part of the problem, the number of solutions $(x, y) \pmod{5^4}$ to $xy(x+y) \equiv 0 \pmod{5^4}$. We will use casework on the number of factors of 5 that divide x, which we will denote by $v_5(x)$.

Case 1: $x \equiv 0 \pmod{5^4}$. Clearly here $y \pmod{5^4}$ can be any value for a total of 5^4 pairs in this case.

Case 2: $v_5(x) = 3$. Note that from 0 to 624 there are 5 numbers divisible by 125 but as we cannot have x with $x \equiv 0 \pmod{5^4}$, there are $5-1=4$ possible x in this case. This implies that $y(y+x) \equiv y^2 \equiv 0 \pmod 5$. This has one solution $\pmod 5$ or 125 solutions $\pmod{625}$ for each x, for a total of $4 * 125$ in this case.

Case 3: $v_5(x) = 2$. Like in case 2, we can similarly count $5^2 - 5 = 20$ possible x in this case. This implies that $y(y+x) \equiv 0 \pmod{25}$. WLOG assume $x = 25$ and note this is $y(y+25) \equiv y^2 \equiv 0 \pmod{25}$. Thus $5|y$ and there are 125 such y, for $20 * 125$ pairs in this case.

Case 4: $v_5(x) = 1$. Like before, we can similarly count $5^3 - 5^2 = 100$ possible x in this case. This implies that $y(y+x) \equiv 0 \pmod{125}$. WLOG assume $x = 5$ and note this is $y(y+5) \equiv 0 \pmod{125}$. This is quite tricky, but as long as $y \equiv 0 \pmod{25}$ or $y \equiv -5 \pmod{25}$ this works, for a total of 10 solutions $y \pmod{125}$ and 50 solutions $\pmod{625}$. We get $100 * 50$ possible pairs in this case.

Case 5: $v_5(x) = 0$. Like before, we can similarly count $5^4 - 5^3 = 500$ possible x in this case. This implies that $y(y+x) \equiv 0 \pmod{625}$, which clearly has two solutions for y, for a total of $500 * 2$ possible pairs in this case.

Adding our cases, we get $5^4 + 4 * 125 + 20 * 125 + 100 * 50 + 500 * 2 = 5^3 * 7 * 11$ possible pairs $(x, y) \pmod{5^4}$. By CRT, there are $5^3 * 7^2 * 11 * 19$ solutions $\pmod{13125}$. As each possible value $\pmod{13125}$ has three equivalent values $\pmod{39375}$, we can multiply by 9 to get $N = 3^2 * 5^3 * 7^2 * 11 * 19$. It follows that the answer is $3 * 4 * 3 * 2 * 2 = \boxed{\textbf{(D) } 144}$.