2$^{\text{nd}}$ Online Test Junior Mathematical Olympiad   (Day **1**)

J-1. Find all functions f taking real numbers to positive integers, such that

$$f^{f(x)}(y) = f(x)f(y)$$

holds true for all real numbers x and y, where $f^a(b)$ denotes the result of a iterations of f on b; i.e. $f^1(b) = f(b)$ and $f^{a+1}(b) = f(f^a(b))$.

J-2. In triangle ABC with circumcircle Γ, let ℓ_1, ℓ_2, and ℓ_3 be the tangents to Γ at points A, B, and C, respectively. Choose a variable point P on side $\overline{BC}$. Let the lines parallel to ℓ_2 and ℓ_3, passing through P, meet ℓ_1 at points C_1 and B_1, respectively. Let the circumcircles of $\triangle PBB_1$ and $\triangle PCC_1$ meet each other again at a point $Q \neq P$. Let lines ℓ_1 and BC meet at a point R, and let lines ℓ_2 and ℓ_3 meet at a point X. Prove that, as P varies on side $\overline{BC}$, lines PQ and RX meet at a fixed point.

J-3. For a positive integer n, let $A_1, A_2, \ldots, A_n$ be distinct subsets of $\{1, 2, \ldots, n+1\}$, each of size at most two. Prove that there exist distinct subsets $\mathcal{S}$ and $\mathcal{S}'$ of $\{1, 2, \ldots, n+1\}$ such that

$$|A_k \cap \mathcal{S}| = |A_k \cap \mathcal{S}'|$$

for all integers $1 \leq k \leq n$, where $|T|$ denotes the number of elements in a set T.

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.