December 2 — December 30

2^{nd} Online Test Junior Mathematical Olympiad (Day 2)

J-4. A $n \times n$ square grid is composed of n^2 unit squares, for a positive integer n. For each unit square in the grid, all of its sides are drawn, and some diagonals of some unit squares are also drawn, so that no unit square has both diagonals drawn and no two unit squares that share a side have diagonals drawn in the same direction. Find all values of n for which there exists a grid configuration such that it is possible to move along a drawn side or diagonal one at a time, starting at the bottom-left vertex of the grid and traversing each segment exactly once.

J-5. Call a positive integer m *cool* if there exists a polynomial $P(x)$ with integer coefficients such that $(P(x))^m - x$ is divisible by m for all positive integers x.

(i) Prove that all cool numbers are square-free.

(ii) Find all positive integers n such that, if $\mathcal{P}_n$ is the product of all primes p such that $n \le p \le 2n$, then $\mathcal{P}_n$ is cool.

Note. A square-free number is an integer which is not divisible by the square of any prime.

J-6. Let ABC be a triangle with circumcenter O, incenter I, and circumcircle Γ. Let there be a circle touching $\overline{AB}$ and $\overline{AC}$, and tangent to Γ internally at a point X. The perpendicular bisector of $\overline{BC}$ meets line AX at a point S. Additionally, let K be the point on the circumcircle of $\triangle AIX$, distinct from I, such that $\overline{KI} \parallel \overline{BC}$. Line KS meets the circumcircle of $\triangle AIX$ again at T. Prove that the tangent at T to the circumcircle of $\triangle TBC$ passes through the circumcenter of $\triangle TAO$.

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.