OTJMO

December 2 — December 30

# 2nd Online Test Junior Mathematical Olympiad (Day 1)

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**J-1.** Find all functions $f$ taking real numbers to positive integers, such that

$$f^{f(x)}(y) = f(x)f(y)$$

holds true for all real numbers $x$ and $y$, where $f^a(b)$ denotes the result of $a$ iterations of $f$ on $b$; i.e. $f^1(b) = f(b)$ and $f^{a+1}(b) = f(f^a(b))$.

**J-2.** In triangle $ABC$ with circumcircle $\Gamma$, let $\ell_1$, $\ell_2$, and $\ell_3$ be the tangents to $\Gamma$ at points $A$, $B$, and $C$, respectively. Choose a variable point $P$ on side $\overline{BC}$. Let the lines parallel to $\ell_2$ and $\ell_3$, passing through $P$, meet $\ell_1$ at points $C_1$ and $B_1$, respectively. Let the circumcircles of $\triangle PBB_1$ and $\triangle PCC_1$ meet each other again at a point $Q \neq P$. Let lines $\ell_1$ and $BC$ meet at a point $R$, and let lines $\ell_2$ and $\ell_3$ meet at a point $X$. Prove that, as $P$ varies on side $\overline{BC}$, lines $PQ$ and $RX$ meet at a fixed point.

**J-3.** For a positive integer $n$, let $A_1, A_2, \ldots, A_n$ be distinct subsets of $\{1, 2, \ldots, n+1\}$, each of size at most two. Prove that there exist distinct subsets $\mathcal{S}$ and $\mathcal{S}'$ of $\{1, 2, \ldots, n+1\}$ such that

$$|A_k \cap \mathcal{S}| = |A_k \cap \mathcal{S}'|$$

for all integers $1 \leq k \leq n$, where $|T|$ denotes the number of elements in a set $T$.

*Time: 4 hours and 30 minutes.*
*Each problem is worth 7 points.*

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**OTSS Committee**
*Wishes you Good Luck!*

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**J-4.** A $n \times n$ square grid is composed of $n^2$ unit squares, for a positive integer $n$. For each unit square in the grid, all of its sides are drawn, and some diagonals of some unit squares are also drawn, so that no unit square has both diagonals drawn and no two unit squares that share a side have diagonals drawn in the same direction. Find all values of $n$ for which there exists a grid configuration such that it is possible to move along a drawn side or diagonal one at a time, starting at the bottom-left vertex of the grid and traversing each segment exactly once.

**J-5.** Call a positive integer $m$ *cool* if there exists a polynomial $P(x)$ with integer coefficients such that $(P(x))^m - x$ is divisible by $m$ for all positive integers $x$.

(i) Prove that all cool numbers are square-free.

(ii) Find all positive integers $n$ such that, if $\mathcal{P}_n$ is the product of all primes $p$ such that $n \leq p \leq 2n$, then $\mathcal{P}_n$ is cool.

*Note.* A square-free number is an integer which is not divisible by the square of any prime.

**J-6.** Let $ABC$ be a triangle with circumcenter $O$, incenter $I$, and circumcircle $\Gamma$. Let there be a circle touching $\overline{AB}$ and $\overline{AC}$, and tangent to $\Gamma$ internally at a point $X$. The perpendicular bisector of $\overline{BC}$ meets line $AX$ at a point $S$. Additionally, let $K$ be the point on the circumcircle of $\triangle AIX$, distinct from $I$, such that $\overline{KI} \parallel \overline{BC}$. Line $KS$ meets the circumcircle of $\triangle AIX$ again at $T$. Prove that the tangent at $T$ to the circumcircle of $\triangle TBC$ passes through the circumcenter of $\triangle TAO$.

*Time: 4 hours and 30 minutes.*
*Each problem is worth 7 points.*

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**OTSS Committee**

*Wishes you Good Luck!*