Online Test Junior Mathematical Olympiad
Season 2 Solutions | OTJMO 2020-21

OTSS Committee

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December 2 — December 30

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§0.1 Day 1 Problems

J-1. Find all functions f taking real numbers to positive integers, such that

$$f^{f(x)}(y) = f(x)f(y)$$

holds true for all real numbers x and y, where $f^a(b)$ denotes the result of a iterations of f on b; i.e. $f^1(b) = f(b)$ and $f^{a+1}(b) = f(f^a(b))$.

J-2. In triangle ABC with circumcircle Γ, let ℓ_1, ℓ_2, and ℓ_3 be the tangents to Γ at points A, B, and C, respectively. Choose a variable point P on side $\overline{BC}$. Let the lines parallel to ℓ_2 and ℓ_3, passing through P, meet ℓ_1 at points C_1 and B_1, respectively. Let the circumcircles of $\triangle PBB_1$ and $\triangle PCC_1$ meet each other again at a point $Q \neq P$. Let lines ℓ_1 and BC meet at a point R, and let lines ℓ_2 and ℓ_3 meet at a point X. Prove that, as P varies on side $\overline{BC}$, lines PQ and RX meet at a fixed point.

J-3. For a positive integer n, let $A_1, A_2, \dots, A_n$ be distinct subsets of $\{1, 2, \dots, n+1\}$, each of size at most two. Prove that there exist distinct subsets $\mathcal{S}$ and $\mathcal{S}'$ of $\{1, 2, \dots, n+1\}$ such that

$$|A_k \cap \mathcal{S}| = |A_k \cap \mathcal{S}'|$$

for all integers $1 \leq k \leq n$, where $|T|$ denotes the number of elements in a set T.

§0.2 Day 2 Problems

J-4. A $n \times n$ square grid is composed of n^2 unit squares, for a positive integer n. For each unit square in the grid, all of its sides are drawn, and some diagonals of some unit squares are also drawn, so that no unit square has both diagonals drawn and no two unit squares that share a side have diagonals drawn in the same direction. Find all values of n for which there exists a grid configuration such that it is possible to move along a drawn side or diagonal one at a time, starting at the bottom-left vertex of the grid and traversing each segment exactly once.

J-5. Call a positive integer m *cool* if there exists a polynomial $P(x)$ with integer coefficients such that $(P(x))^m - x$ is divisible by m for all positive integers x.

(i) Prove that all cool numbers are square-free.

(ii) Find all positive integers n such that, if $\mathcal{P}_n$ is the product of all primes p such that $n \leq p \leq 2n$, then $\mathcal{P}_n$ is cool.

Note. A square-free number is an integer which is not divisible by the square of any prime.

J-6. Let ABC be a triangle with circumcenter O, incenter I, and circumcircle Γ. Let there be a circle touching $\overline{AB}$ and $\overline{AC}$, and tangent to Γ internally at a point X. The perpendicular bisector of $\overline{BC}$ meets line AX at a point S. Additionally, let K be the point on the circumcircle of $\triangle AIX$, distinct from I, such that $\overline{KI} \parallel \overline{BC}$. Line KS meets the circumcircle of $\triangle AIX$ again at T. Prove that the tangent at T to the circumcircle of $\triangle TBC$ passes through the circumcenter of $\triangle TAO$.

§1 Day 1 Solutions

§1.1 Solution to J-1, proposed by NJOY

Find all functions f taking real numbers to positive integers, such that

$$f^{f(x)}(y) = f(x)f(y)$$

holds true for all real numbers x and y, where $f^a(b)$ denotes the result of a iterations of f on b; i.e. $f^1(b) = f(b)$ and $f^{a+1}(b) = f(f^a(b))$.

Answer. $f \equiv 1$ is the only such function. It is easy to check it indeed satisfies the equation.

Notations: The symbol $\mathbb{N}$ denotes the set of positive integers.

Solution 1 (by tastymath75025) Let S be the range of f. Then, by the given equation,

$$f^{P-1}(Q) = PQ,$$

for all $P, Q \in S$. But, since $f^{f(x)}(x) = f^2(x)$, if $P \in S \Rightarrow P^2 \in S$. Hence, $f^{P^2-1}(Q) = P^2Q$ as well. Now we prove the following Claim:

Claim 1.1.1. $f^{x(P-1)}(Q) = P^xQ$, for all $P, Q \in S$ and $x \in \mathbb{N}$.

Proof. We prove it using induction. The base case $x = 1$ is already proven. Suppose that the claimed relation holds for $x = k$. Then, for $x = k + 1$,

$$f^{(x+1)(P-1)}(Q) = f^{x(P-1)}(f^{P-1}(Q)) = f^{x(P-1)}(PQ) = P^{x+1}Q,$$

where the last equality follows from the $x = k$ case of the claimed relation and the fact that if $P, Q \in S \Rightarrow PQ \in S$. □

Back to the problem, taking $x \to P + 1$,

$$f^{P^2-1}(Q) = P^{P+1}Q,$$

so combining this with $f^{P^2-1}(Q) = P^2Q$, we obtain

$$P^{P+1} = P^2,$$

for all $P \in S$. Hence, $P^{P-1} = 1$, and since $P \in \mathbb{N}$, we obtain $P = 1$. Therefore, $f \equiv 1$, which obviously satisfies, so it is our only solution. ■

Solution 2 (by Orestis Lignos) Let $P(x, y)$ denote the assertion

$$P(x, y) : f^{f(x)}(y) = f(x)f(y).$$

We prove the following Claims:

Claim 1.1.2. $f(f(x)) = cf(x)$ for a constant c and for every $x \in \mathbb{R}$.

Proof. Note that, $P(x, f(y))$ yields

$$f(x)f(f(y)) = f^{f(x)}(f(y)) = f(f^{f(x)}(y)) = f(f(x)f(y)).$$

Now, by switching x, y, we obtain $f(f(x))f(y) = f(f(x)f(y))$. Combining both relations yield

$$\frac{f(f(x))}{f(x)} = \frac{f(f(y))}{f(y)}. \tag{1.1}$$

Note that 1.1 holds true for any $x, y \in \mathbb{R}$. So, the ratio is equal to a constant, say c. Therefore, $f(f(x)) = cf(x)$ holds true for any $x \in \mathbb{R}$ and for some constant c, as claimed. □

Claim 1.1.3. $f^n(x) = c^{n-1}f(x)$, for all positive integers n and for all $x \in \mathbb{R}$.

Proof. We prove the Claim using induction. Indeed, the base case $n = 1$ is obvious. Suppose that the claimed relation holds for $n = k$. Then, for $n = k + 1$,

$$f^{k+1}(x) = f^k(f(x)) = c^{k-1} \cdot f(f(x)) = c^{k-1} \cdot cf(x) = c^k f(x),$$

completing the inductive step and hence we conclude this claim due to induction. □

Now, by using the result of Claim 1.1.3,

$$f(x)f(y) = f^{f(x)}(y) = c^{f(x)-1} \cdot f(y) \implies c^{f(x)-1} = f(x),$$

for all $x \in \mathbb{R}$. Let $f(x) = m \in \mathbb{N}$ (m is variable) and $c = \dfrac{A}{B}$ with $\gcd(A, B) = 1$ and $A, B \in \mathbb{N}$. Then, $c^{f(x)-1} = f(x)$ rewrites as $A^{m-1} = mB^{m-1}$.

Since $B^{m-1} \mid A^{m-1}$ and $\gcd(A, B) = 1$, we obtain $B = 1$, hence $A^{m-1} = m$. If $A = 1$, then $m = 1$, hence $f \equiv 1$, which satisfies. Suppose that $A \geq 2$. Then,

$$m = A^{m-1} \geq 2^{m-1}.$$

But, $2^{m-1} > m$ if $m \geq 3$ using easy induction, hence $m \in \{1, 2\}$, that is $f(x) \in \{1, 2\}$ for all x. Suppose that there exists a t such that $f(t) = 2$. Then, $c^{f(t)-1} = f(t)$, hence $c = 2$, implying

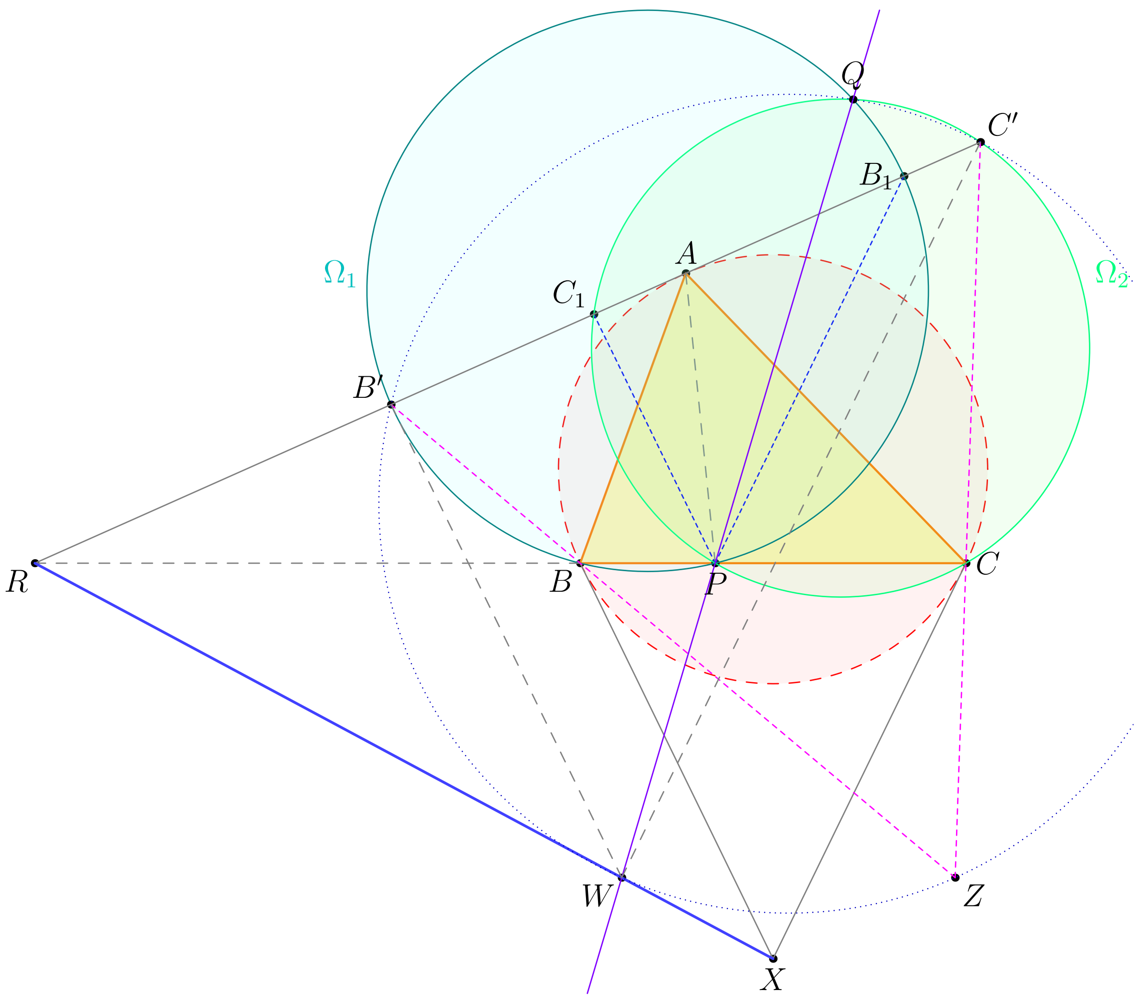
$$4 = 2f(t) = cf(t) = f(f(t)) = f(2) \in \{1, 2\},$$

a contradiction. Therefore, $f(x) = 1$, for all x, giving that f is constantly equal to 1.

Hence, we conclude that $f \equiv 1$ is the only such function satisfying the given equation. ■

§1.2 Solution to J-2, proposed by NJOY & Orestis_Lignos

In triangle ABC with circumcircle Γ, let ℓ_1, ℓ_2, and ℓ_3 be the tangents to Γ at points A, B, and C, respectively. Choose a variable point P on side $\overline{BC}$. Let the lines parallel to ℓ_2 and ℓ_3, passing through P, meet ℓ_1 at points C_1 and B_1, respectively. Let the circumcircles of $\triangle PBB_1$ and $\triangle PCC_1$ meet each other again at a point $Q \neq P$. Let lines ℓ_1 and BC meet at a point R, and let lines ℓ_2 and ℓ_3 meet at a point X. Prove that, as P varies on side $\overline{BC}$, lines PQ and RX meet at a fixed point.



Remark. Although the problem admits a pure angle-chase solution and power of a point solution, it is not easy at all. It is pretty difficult to identify the fixed point, without the aid of Geogebra/Asymptote etc. Several testsolvers pointed out to us that it may be even be too difficult for its position.

Let XYZ be the triangle formed by ℓ_1, ℓ_2, and ℓ_3, where Y lies on ℓ_1 and ℓ_3, and Z lies on ℓ_1 and ℓ_2. Let the circumcircles of $\triangle PBB_1$ and $\triangle PCC_1$ intersect line YZ at points $B' \neq B_1$ and $C' \neq C_1$, respectively.

First Step (Identifying the fixed point) We begin by proving a claim.

Claim 1.2.1. Points B' and C' are fixed, regardless of the position of P.

Proof. Note that $\angle ABB' = \angle B'BP - \angle B = \angle PB_1Y - \angle B = 180^\circ - \angle AYC - \angle B = \angle B$, and since B' lies on ℓ_1, it is fixed. Analogously, we obtain that C' is fixed as well. $\square$

Remark. On a side note, points B' and C' are such that line AB is tangent to the circumcircle of $\triangle ABB'$ and line AC is tangent to the circumcircle of $\triangle ACC'$.

Now, let $W \neq Q$ be the intersection of line PQ with the circumcircle of triangle $QB'C'$.

Claim 1.2.2. W is the desired fixed point.

Proof. Indeed, note that $\angle WB'C' = \angle WQC' = \angle PQC' = 180^\circ - \angle PCC' = 180^\circ - 2\angle C = \angle XZY$, hence $\overline{WB'} \parallel \overline{XZ}$. Similarly, $\overline{XY} \parallel \overline{WC'}$, which implies that W is the point of intersection of the parallels from B' to line XZ and from C' to line XY, which is fixed, using Claim 1.2.1. $\square$

Second Step (Fixed point lies on RX)

Method 1 (straightforward bash)

Let lines XW, ℓ_1 meet at point R'. Then, using $B'W \parallel XZ$ and $WC' \parallel XY$, we obtain

$$\frac{R'B'}{R'Z} = \frac{R'W}{R'X} = \frac{R'C'}{R'Y},$$

hence $\dfrac{R'B'}{R'Z} = \dfrac{R'C'}{R'Y}$.

Claim 1.2.3. Only one point E exists on ray $\overline{YTSZ}$ such that $\dfrac{EB'}{EZ} = \dfrac{EC'}{EY}$.

Proof. Note that

$$\frac{EB'}{EZ} = \frac{EC'}{EY} \Rightarrow \frac{EB'}{ZB'} = \frac{EC'}{C'Y},$$

hence

$$\frac{EB'}{EC'} = \frac{B'Z}{C'Y} \Rightarrow \frac{EB'}{B'C'} = \frac{B'Z}{C'Y - B'Z},$$

so

$$EB' = \frac{B'C' \cdot B'Z}{C'Y - B'Z},$$

which is constant, therefore E is unique. $\square$

From the Claim, it suffices to show that $\dfrac{RB'}{RZ} = \dfrac{RC'}{RY}$, or equivalently $\dfrac{RB'}{RC'} = \dfrac{RZ}{RY}$, since this would imply that $R \equiv R'$ hence line XW passes through R, as desired.

Since the circumcircle of triangle ABC is the incircle of triangle XYZ, we know that lines XA, YB, ZC are conccurent. An easy proof of this fact follows by Ceva's Theorem. From the complete quadrilateral $XBAC.ZY$, we obtain that $(R, Z, A, Y) = -1$, hence $\dfrac{RZ}{RY} = \dfrac{AZ}{AY}$.

In the following trig bash, trivial computation of angles is omitted. We know that $\angle B'BA = \angle B, \angle ACC' = \angle C$. Employing the Law of Sines, we obtain:

- in triangle $RB'B$, $\dfrac{RB'}{\sin 2B} = \dfrac{B'B}{\sin(B - C)}$,

- in triangle $RC'C$, $\dfrac{RC'}{\sin 2C} = \dfrac{C'C}{\sin(B-C)}$.

So by dividing the previous two relations, $\dfrac{RB'}{RC'} = \dfrac{\sin 2B}{\sin 2C} \cdot \dfrac{B'B}{C'C}$. Again by Law of Sines,

- in triangle $B'BA$, $\dfrac{B'B}{\sin C} = \dfrac{AB}{\sin A}$,
- in triangle $C'CA$, $\dfrac{C'C}{\sin B} = \dfrac{AC}{\sin A}$.

Dividing the two relations, $\dfrac{B'B}{C'C} = \dfrac{\sin^2 C}{\sin^2 B}$, since $\dfrac{AB}{AC} = \dfrac{\sin C}{\sin B}$. Hence, $\dfrac{RB'}{RC'} = \dfrac{\sin^2 C}{\sin 2C} \cdot \dfrac{\sin^2 B}{\sin 2B}$.

To finish, note that, by Sine Law:

- in triangle ZAB, $\dfrac{AZ}{\sin C} = \dfrac{AB}{\sin 2C}$,
- in triangle YAC, $\dfrac{AY}{\sin B} = \dfrac{AC}{\sin 2B}$.

After dividing these two relations and using that $\dfrac{AB}{AC} = \dfrac{\sin C}{\sin B}$, we obtain that

$$\frac{AZ}{AY} = \frac{\sin^2 C}{\sin 2C} \cdot \frac{\sin^2 B}{\sin 2B} = \frac{RB'}{RC'},$$

hence we are done. ■

Method 2 (much shorter ending of Method 1)

In this solution, we present an alternative way of proving the relation $\dfrac{RB'}{RZ} = \dfrac{RC'}{RY}$ of **Method 1**. Note that

$$\angle BZC' = 180^\circ - 2\angle C = 180^\circ - \angle BCC',$$

so $BZC'C$ is cyclic. Similarly, $BB'YC$ is cyclic.
Hence, $RZ \cdot RC' = RB \cdot RC = RB' \cdot RY$, which rewrites to the desired relation.

> **Remark.** Basically, Method 2 provides a non-trigonometric way to prove the relation $\dfrac{RB'}{RZ} = \dfrac{RC'}{RY}$. Although it is pretty short, it is not so easy to find.

§1.3 Solution to J-3, proposed by Supercali

For a positive integer n, let $A_1, A_2, \ldots, A_n$ be distinct subsets of $\{1, 2, \ldots, n+1\}$, each of size at most two. Prove that there exist distinct subsets $\mathcal{S}$ and $\mathcal{S}'$ of $\{1, 2, \ldots, n+1\}$ such that

$$|A_k \cap \mathcal{S}| = |A_k \cap \mathcal{S}'|$$

for all integers $1 \le k \le n$, where $|T|$ denotes the number of elements in a set T.

Solution (a1267ab). So we're proving this claim by induction:
If $A_1, \ldots, A_n$ are subsets of $\{1, 2, \ldots, n+1\}$ of size at most 2, then there exists distinct subsets I, J such that $|A_k \cap I| = |A_k \cap J|$ for all $k \le n$.
Suppose one of those sets is a singleton, say $A_n = \{n+1\}$. By the inductive hypothesis, there exist subsets I', J' of $\{1, 2, \ldots, n\}$ such that $|A_k \cap I'| = |A_k \cap J'|$ for all $k \le n-1$. (Note that it doesn't matter if $n+1 \in A_k$.) Then we can take $I = I', J = J'$.
Induct on n. Remove all singleton sets and the associated elements: this reduces to a smaller problem, so assume that all sets have size 2. Then we can view these as a collection of n edges on $n+1$ vertices. At least one connected component must be a tree. Then 2-color that tree to obtain I, J. ■

Solution (outline, by **p_square**). Consider all multisets of multiplicity at most 2 from $\{1, 2, \ldots n+1\}$. For each such multiset consider the vector $|A_k \cap I| \pmod 3$. There are 3^n such vectors, but 3^{n+1} such mulitsets. Hence for two distinct ones I, J the vectors are the same. Now if an element occurs twice in I and at most once in J then delete both occurrences from I and add one occurrence to J. If an element occurs twice in both I, J, delete all 4 occurrences. Thus we obtain sets I', J' for which the vectors are the same. But $|A_k \cap I| \le 2$ anyway so in fact the sizes are equal. ■

§2 Day 2 Solutions

§2.1 Solution to J-4, proposed by ARMLlegend & reaganchoi

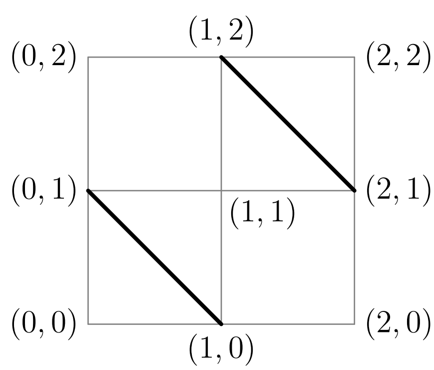
A $n \times n$ square grid is composed of n^2 unit squares, for a positive integer n. For each unit square in the grid, all of its sides are drawn, and some diagonals of some unit squares are also drawn, so that no unit square has both diagonals drawn and no two unit squares that share a side have diagonals drawn in the same direction. Find all values of n for which there exists a grid configuration such that it is possible to move along a drawn side or diagonal one at a time, starting at the bottom-left vertex of the grid and traversing each segment exactly once.

Answer. The answer is $n \in \{1, 2, 3\}$.

For the sake of convenience, we put the grid in the coordinate plane, with the bottom-left vertex of the grid at the origin.

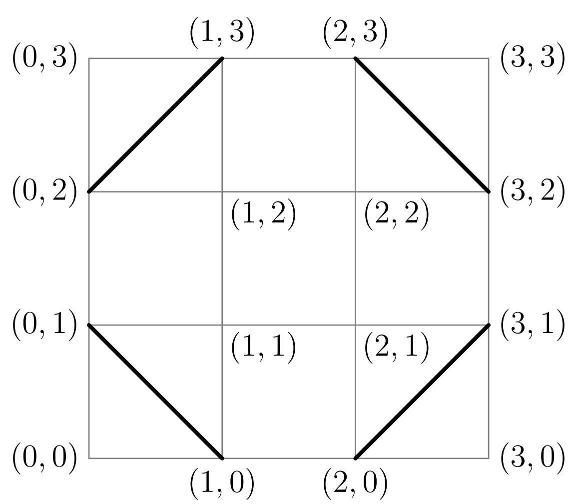
Case 1. $n = 1$. We can easily see that this case satisfies the conditions.

Case 2. $n = 2$. We can satisfy the conditions by connecting two opposite diagonals that "cut off a corner".



For example, we can draw the diagonal connecting $(0, 1)$ and $(1, 0)$ as well as the diagonal connecting $(1, 2)$ and $(2, 1)$.

Case 3. $n = 3$. We can satisfy the conditions by connecting all four diagonals that "cut off a corner".



Case 4. $n > 3$. We claim that at least one edge vertex on each side will end up with at least one vertex with odd degree.

Assume that some side of the square has only vertices with even degree (4). Then, consider the center vertex of the side, which we will call $(a, 0)$. There must be exactly one diagonal connecting it, let us say from $(a, 0)$ to $(a + 1, 1)$.

Then, the vertex $(a + 1, 0)$ also needs a diagonal, but both possible diagonals are invalid by the restrictions, a contradiction.

In conclusion, we have found that the only n which satisfy the problem condition are $n \in \{1, 2, 3\}$. ■

§2.2 Solution to J-5, proposed by Orestis_Lignos

Call a positive integer m *cool* if there exists a polynomial $P(x)$ with integer coefficients such that $(P(x))^m - x$ is divisible by m for all positive integers x.

(i) Prove that all cool numbers are square-free.

(ii) Find all positive integers n such that, if $\mathcal{P}_n$ is the product of all primes p such that $n \le p \le 2n$, then $\mathcal{P}_n$ is cool.

Note. A square-free number is an integer which is not divisible by the square of any prime.

Answer. (ii): All $n \neq 2$ satisfy the problem condition.

(i) Since $m \mid P(x)^m - x$, we obtain that $t^m \pmod m$ must contain all possible residues. So, this means that the set $\{1^m, 2^m, \dots, m^m\}$ is complete $\pmod m$. Since this set and $\{0, 1, \cdots, m-1\}$ have both cardinality m, any i^m corresponds to exactly one remainder from the set $\{0, 1, \cdots, m-1\}$. Hence, $i^m \neq j^m \pmod m$ for all $i \neq j$ with $i, j \le m$.

Suppose now that m is not square-free. Then $m = {p_1}^{k_1} \cdots {p_r}^{k_r}$ with p_i primes and at least one of k_i being > 1. Consider now $s = p_1 p_2 \cdots p_r$. Since there exists a $k_i > 1$, we conclude that $m > s$.

In addition, $s^m = p_1^m \cdots p_r^m$. If there existed a i such that $k_i \ge m$, then

$$m \ge {p_i}^{k_i} \ge 2^m \ge m + 1,$$

a contradiction. So $m > k_i$ for all i, which means that

$${p_1}^{k_1} \cdots {p_r}^{k_r} \mid s^m = {p_1}^m \cdots {p_r}^m.$$

Therefore, $s^m \equiv m^m \pmod m$, and $s < m$ as established before. This is a contradiction. Hence, m must be square-free. ■

(ii) Let $\mathcal{P}_n = p_1 p_2 \dots p_k$ with p_i being all primes between n and $2n$, and $p_1 < p_2 < \dots < p_k$. We distinguish some cases:

Case 1. $n = 1$. Then, $\mathcal{P}_n = 2$, so by taking $P(x) = x$, we have that $2 = \mathcal{P}_n \mid x^2 - x$.

Case 2. $n = 2$. Then, $\mathcal{P}_n = 6$. If 6 was cool, then there would exist a $P(x)$ such that $6 \mid P(x)^6 - x$ for all $x \in \mathbb{Z}$, a contradiction since if $x \equiv 2 \pmod 3$, then $P(x)^6 \equiv 2 \pmod 3$, which is not possible.

Case 3. $n > 2$. We will prove that $\mathcal{P}_n$ is cool. In order to do so, we aim to pick $P(x) = x^t$ for suitable t. The given condition rewrites as $p_i \mid x^{tp_1 p_2 \cdots p_k} - x$ for all i.

By Fermat's Little Theorem, $x_i^p - 1 \equiv 1 \pmod{p_i}$ if $p_i \nmid x$, so if we pick t such that $(p_i - 1) \mid (tp_1 \cdots p_r - 1)$, then we would have:

$$x^{tp_1 \cdots p_k} = x^{s(p_i - 1) + 1} = x \cdot (x^{p_i - 1})^s \equiv x \pmod{p_1},$$

hence we have the desired.

What it remains now, is to pick t such that $(p_i - 1) \mid (tp_1 \cdot p_r - 1)$ for all i, or equivalently

$$\operatorname{lcm}(p_1 - 1, \cdots p_r - 1) \mid (tp_1 \cdots p_r - 1).$$

We now make the following Claim.

Claim 2.2.1. We have $\gcd(\operatorname{lcm}(p_1 - 1, \cdots p_r - 1), tp_1 \cdots p_r - 1) = 1$.

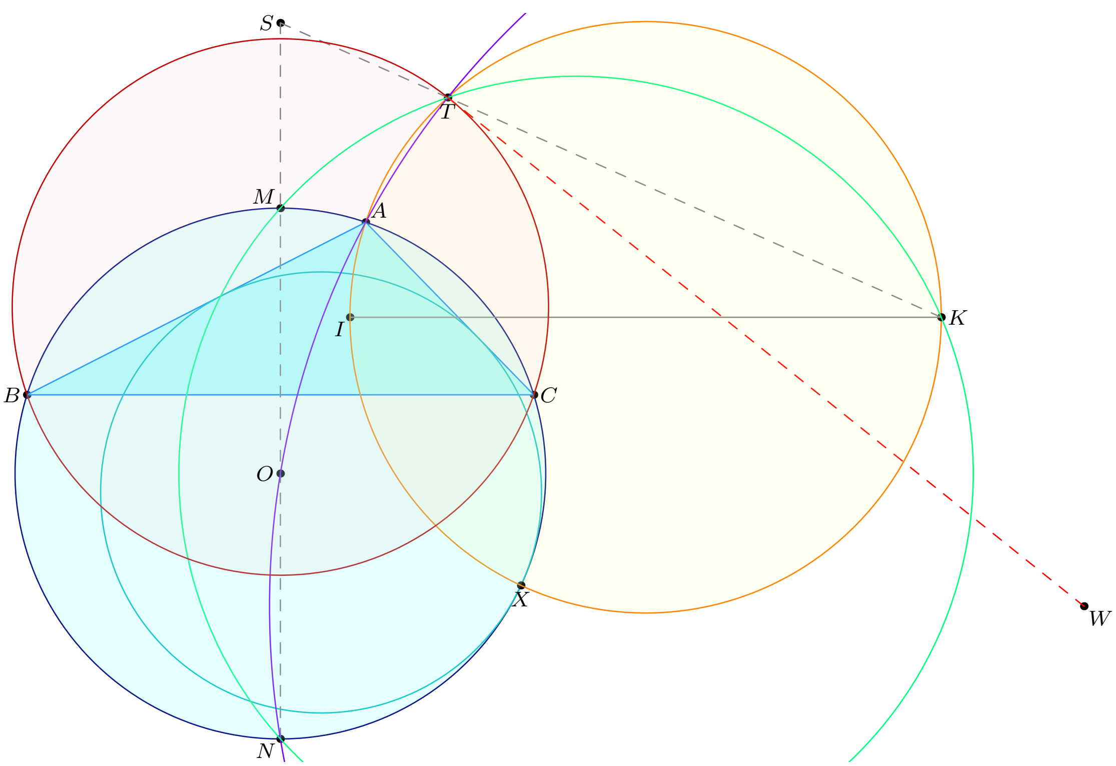
Proof. Suppose otherwise. Then, there must exist indices i, j such that $p_i \mid (p_j - 1)$. If $p_j - 1 \geq 2p_i$, we obtain $2n > p_k - 1 \geq p_j - 1 \geq 2p_i \geq 2p_1 \geq 2n$, a contradiction. Therefore $p_j - 1 = p_i$ which means $p_j = 3, p_i = 2$ since they are primes. But this is a clear contradiction, since $p_i \geq n > 2$, for all i. □

Back to the problem, from the Claim's result we finish easily by taking as t the inverse of $p_1 \ldots p_r$ modulo $\operatorname{lcm}(p_1 - 1, \cdots p_r - 1)$.

Hence, we conclude that all $n \neq 2$ satisfy the problem condition. ■

§2.3 Solution to J-6, proposed by Orestis_Lignos

Let ABC be a triangle with circumcenter O, incenter I, and circumcircle Γ. Let there be a circle touching $\overline{AB}$ and $\overline{AC}$, and tangent to Γ internally at a point X. The perpendicular bisector of $\overline{BC}$ meets line AX at a point S. Additionally, let K be the point on the circumcircle of $\triangle AIX$, distinct from I, such that $\overline{KI} \parallel \overline{BC}$. Line KS meets the circumcircle of $\triangle AIX$ again at T. Prove that the tangent at T to the circumcircle of $\triangle TBC$ passes through the circumcenter of $\triangle TAO$.



Solution 1, by proposer. Let M, N be the points where the perpendicular bisector of BC intersects (Γ), where M belongs to the arc BAC. Then, $SM \cdot SN = SA \cdot SX = ST \cdot SK$, hence $KTMN$ is cyclic. Let I be the incenter, and TI intersect MN at O'. We claim that $O' \equiv O$. We prove the following known Claim:

Claim 2.3.1. Points X, I, M are collinear.

Proof. Let Y, Z be the tangent points of circle (γ) with AB, AC. The homothety with center X sends Y to the midpoint P of the arc AB, hence X, Y, P are collinear, and similarly X, Z, Q are collinear with Q being the midpoint of the arc AC. By Pascal's theorem on $PCXABQ$, we have that Y, Z and $I \equiv CP \cap BQ$ are collinear. So, $\angle YIB = \angle AIB - 90^\circ = \angle C/2 = \angle YXB$, hence $YBXI$ is cyclic. Thus, $\angle BXI = \angle AYI = 90^\circ - \angle A/2 = \angle BXM$, implying that X, I, M are collinear. □

To the problem, since

$$\angle AKI = \angle AXI = \angle AXM,$$

and

$$\angle KIA = \angle C + \angle A/2 = 90^\circ - \angle AXM,$$

we have that

$$\angle KAI = 180^\circ - \angle AKI - \angle AIK = 90^\circ.$$

In addition A, I, N and M, I, X are collinear, hence I is the orthocenter of KMN.
Now,

$$\angle MTO' = \angle MTK - 90^\circ = 180^\circ - \angle MNK - 90^\circ - 90^\circ - \angle MNK = \angle IMO',$$

so $\angle MTO' = \angle IMO'$, which implies that $O'M^2 = O'I \cdot O'T$, and similarly $O'N^2 = O'I \cdot O'T$. Therefore, $O'M = O'N$, implying that $O' \equiv O$.
Let P be the circumcenter of triangle TBC. We need to show that $\angle WTP = 90^\circ$, or equivalently that $\angle ATP = 90^\circ - \angle ATW$. But,

$$\angle ATW = 90^\circ - \angle TWA/2 = 90^\circ - \angle TOA,$$

so it suffices to show that $\angle TOA = \angle ATP$.
We prove the following Claims:

Claim 2.3.2. $\angle BTC = |\angle BIC - \angle BOC| = \left|\frac{3\angle A}{2} - 90^\circ\right|$.

Proof. Refer to the attached diagram. Note that $\angle OAI = (\angle B - \angle C)/2 = \angle ATI$ by an easy angle-chase and by assuming WLOG that $\angle B > \angle C$. So $\angle OAI = \angle ATI$, hence OA is tangent to the circumcircle of $KTAI$. Hence $OA^2 = OI \cdot OT$.
So $OB^2 = OC^2 = OI \cdot OT$, implying that $\angle OBI = \angle OTB, \angle OCI = \angle OTC$. Therefore, $\angle BTC = \angle OBI + \angle OCI$.
But now note that by angle-chase $\angle OBI + \angle OCI = 360^\circ - \angle BIC - \angle BOC = |3\angle A/2 - 90^\circ|$. □

Claim 2.3.3. Quadrilateral $TAON$ is cyclic.

Proof. Note that, $\angle ATN = \angle ATO + \angle OTN = \angle AKI + 90^\circ - \angle KTN = (\angle B - \angle C)/2 + 90^\circ - \angle KMN = (\angle B - \angle C)/2 + \angle MKI = \angle B - \angle C = \angle AOM$, hence $\angle ATN = \angle AOM$, so $TAON$ is cyclic. □

To the problem, $\angle PBC = 90^\circ - \angle BTC$, and $\angle OBC = \angle A - 90^\circ$, so

$$\angle PBO = \angle PBC + \angle OBC = 90^\circ - \angle A/2 = \angle PNB,$$

hence $\angle PBO = \angle PNB \Rightarrow PB^2 = PO \cdot PN$, giving $PA^2 = PO \cdot PN$, implying $\angle PTO = \angle ONT$.
By the second Claim,

$$\angle TAO = 180^\circ - \angle ONT = 180^\circ - \angle PTO \Rightarrow \angle PTO = 180^\circ - \angle TAO = \angle ATO + \angle AOT.$$

So, $\angle TOA = \angle PTO - \angle ATO = \angle ATP$, and the proof is complete. ■

Solution 2. (by tastymath75025, inversive) As in **Solution 1**, we obtain that:

(i) Points X, I, M are collinear.

(ii) Points T, I, O are collinear.

(iii) Quadrilateral $TAON$ is cyclic.

Perform an inversion with center O and power OA^2. This inversion sends T to I, since $OA^2 = OI \cdot OT$. So, the circumcircle of triangle TBC goes to the circumcircle of triangle BIC.
In addition, if W is the circumcenter of triangle $TAON$, then $WA = WN = WT$ so $OW \perp AN$. Therefore, if O' is the symmetric point of O with respect to AN, then O' is the image of W under this inversion. Indeed, note that

$$\angle IO'O = \angle IOO' = \angle OTW,$$

therefore $TIO'W$ is cyclic, implying

$$OO' \cdot OW = OI \cdot OT = OA^2.$$

To end, line WT is therefore sent to the circumcircle of triangle OIO'.
So, what it remains to prove is that the circumcircles of triangles

$$OIO', BIC,$$

are tangent. But this is obvious, since both circumcircles are symmetric with respect to line AI. The proof is complete. ■