OTSS TMC

Snowy Mathematics Competitions

Snowy Mathematics Competitions

2nd Annual

TMC 10

Thursday, December 31, 2020

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU TELL YOURSELF.
2. This is a twenty-five question multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. Mark your answer to each problem on a TMC 10 Private Message with a keyboard. Check the keys for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded; however, this mock will be graded by people.
4. Scoring: You will receive **6** points for each correct answer, **1.5** points for each problem left unanswered, and **0** points for each incorrect answer.
5. No aids are permitted other than scratch paper, graph paper, rulers, compass, protractors, and erasers. No calculators, smartwatches, or computing devices are allowed. No problems on the test will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your proctor will not ask you to record certain information on the private message.
8. When you give the signal, begin working on the problems. You will have 75 minutes to complete the test. You can discuss only with people that have taken the test during the period when make-ups are eligible.
9. When you finish the exam, don't sign your name in the space provided on the Private Message.

The Committee on the Test Mathematics Competitions reserves the right to re-examine students before deciding whether to grant official status to their scores. The Committee also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

Students who score well on this TMC 10 will not be invited, but rather encouraged, to take the Season 2 Online Test Invitational Examination (OTIE) from January 8, 2021, to January 22, 2021. More details about the OTIE and other information are not on the back page of this test booklet.

1. In square $ABCD$, let M be the midpoint of side $\overline{CD}$, and let N be the reflection of M over side $\overline{AB}$. What fraction of $\triangle MND$ lies within $ABCD$?

(A) $\frac{1}{2}$ **(B)** $\frac{5}{8}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{3}{4}$ **(E)** $\frac{7}{8}$

2. Jack is reading a 100 page book. He reads two pages every minute. After every 12 pages he reads, he takes a one minute break, and then he goes back to reading. If Jack starts reading at 2:00, what time will it be when he finishes reading his book?

(A) 2:32 **(B)** 2:56 **(C)** 2:58 **(D)** 3:05 **(E)** 3:18

3. Let a be the the largest solution to the equation

$$(x^2 + 6x + 8)(x^2 - 16x + 55) = 0,$$

and let b be the smallest solution. What is $a - b$?

(A) 15 **(B)** 16 **(C)** 17 **(D)** 18 **(E)** 22

4. Pedro currently has 2 quarters, 3 dimes and 2 pennies. If he can only obtain quarters, dimes, nickels, and pennies, what is the minimum number of coins he needs to earn in order to reach a total of exactly 1 dollar?

(A) 2 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 10

5. 15 students are to be randomly split into 5 groups of 3 to work on a project. Alice, Bob, and Cooper are three of the students. Given that Alice and Bob are in the same group, what is the probability that Cooper is not in that group?

(A) $\frac{4}{7}$ **(B)** $\frac{3}{4}$ **(C)** $\frac{4}{5}$ **(D)** $\frac{8}{9}$ **(E)** $\frac{12}{13}$

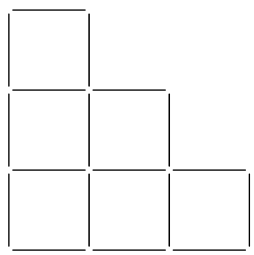
6. Mark wants to distribute all 100 pieces of his candy to his five children, Albert, Bob, Charlie, Diana and Ethan. Diana and Ethan insist on each having a prime number of candies whose sum is also a prime number. Charlie insists on having exactly 35 candies, exactly 1 more than Albert and Bob's amounts combined. Given that Ethan has the smallest number of candies, how many candies must Mark give to Diana?

(A) 3 **(B)** 7 **(C)** 17 **(D)** 23 **(E)** 29

7. When all 9 diagonals of a regular hexagon are drawn, they partition the hexagon into some number of individual regions. What percent of these regions are triangular?

(A) 25% **(B)** 37.5% **(C)** 50% **(D)** 62.5% **(E)** 75%

8. A 3-step staircase is shown below, where each side of each square is of unit length.



Extend this pattern to create two 2021-step staircases. When these two staircases are fit together to form a 2021×2022 rectangle, the two staircases meet each other at a crease of length L. What is L? (In the resulting rectangle, a crease is defined as the total length of the segments touched by both of the original staircases.)

(A) 2022 **(B)** 2023 **(C)** 4041 **(D)** 4042 **(E)** 4043

9. Let the area of equilateral $\triangle ABC$ be 9. Let O denote the center of its circumcircle. Let ω_1 and ω_2 be circles centered at A and B, respectively, such that they both pass through O. Let ω_1 and ω_2 intersect at P, distinct from O. What is the area of $CBPA$?

(A) $6\sqrt{3}$ **(B)** 12 **(C)** $8\sqrt{3}$ **(D)** $9\sqrt{3}$ **(E)** 15

10. In a regular hexagon with side length 2, three of the sides are chosen at random. Next, the midpoints of each of the chosen sides are drawn. What is the probability that the triangle formed by the three midpoints has a perimeter which is an integer?

(A) $\frac{1}{10}$ **(B)** $\frac{1}{5}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{2}{5}$ **(E)** $\frac{1}{2}$

11. A positive integer is called a *flake* if it has at least three distinct prime factors. Two flakes are defined to be in a *snowflake* if there exists a prime that is divisible by the greatest common divisor of the two flakes. When two flakes in a snowflake are multiplied, what is the smallest possible number of divisors in the resulting number?

(A) 18 **(B)** 27 **(C)** 48 **(D)** 54 **(E)** 64

12. A group of people are in a room. It is given that 5 people have a pet dog, 6 people have a pet cat, 8 people have a pet fish, and 3 people have no pets. If no one has more than two pets, and no one has more than one of the same type of pet, what is the smallest possible number of people in the room?

(A) 10 **(B)** 13 **(C)** 14 **(D)** 16 **(E)** 19

13. A function f is defined by a real-valued expression

$$f(x) = \frac{2020x + 1}{x + 2020} + \frac{\sqrt[4]{|x| - 2021} - \sqrt[8]{2021 - |x|}}{|x - 2021|}.$$

For x in the domain of f, $f(x)$ can be simplified to a value N. What is the remainder when N is divided by 100?

(A) 19 **(B)** 21 **(C)** 41 **(D)** 59 **(E)** N is not an integer.

14. For a positive integer n, define a function $f(n)$ to be equal to the largest integer k such that n is divisible by 2^k. For example, $f(8) = 3$ and $f(12) = 2$. Now, let p, q, and r be distinct primes less than 100, so that M is the largest value that

$$f((p + 1)(q + 1)(r + 1))$$

can take, and m is the smallest value. What is $M + m$?

(A) 11 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16

15. A right circular cone has base radius 3 and height 4. Let s be the side length of the largest possible cube that can be inscribed inside the cone such that two of its faces are parallel to the base of the cone. Then $s = a - b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by the square of any prime. What is $a + b + c$?

(A) 62 **(B)** 64 **(C)** 66 **(D)** 68 **(E)** 70

16. Bob and Bill are playing with 7 red cards and 3 blue cards. The cards are numbered from 1 to 10, inclusive, and are flipped over so that neither person can see the numbers. Bill knows that the blue cards are numbered 1, 8, and 9, in some order, but Bob does not know this, nor does he know that Bill knows. Bill chooses a card first, and then Bob chooses a different card. A player wins the game if they choose a card with a larger number. If both players play optimally, what is the probability that Bill wins?

(A) $\frac{1}{2}$ **(B)** $\frac{5}{9}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{7}{10}$ **(E)** $\frac{25}{27}$

17. Farmers A and B are mowing a rectangular grid of 21 rows and 23 columns of cells of grass. A starts from the top-left cell of the grid and mows right until he has mowed all the grass in the row. Then, B starts from where A stopped and mows down the same column until he has mowed all the grass in the column. They keep taking turns mowing, each time turning 90° clockwise, switching the farmer, and mowing only unmowed grass in the same direction until not possible. If A and B mow at rates of 7 and 3 cells per minute, respectively, and it takes no time to turn 90° clockwise and switch farmers, how long, in minutes, will it take them to finish mowing all the grass?

(A) 108 **(B)** 109 **(C)** 110 **(D)** 111 **(E)** 112

18. A sequence is defined recursively by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all integers $n \geq 2$. Let S be the sum of all positive integers n such that $|F_n - n^2| \leq 3n$. What is the sum of the digits of S?

(A) 6 **(B)** 7 **(C)** 10 **(D)** 11 **(E)** 14

19. In triangle ABC, $AB = 15$ and $BC = 20$, with a right angle at B. Point D is chosen on side $\overline{AC}$ and is reflected over sides $\overline{AB}$ and $\overline{BC}$ (not consecutively) to create points M and N, respectively. What is the smallest possible value of the length MN?

(A) 18 **(B)** 21 **(C)** 24 **(D)** 27 **(E)** 30

20. A fair coin is painted such that one side is red and the other side is blue. A fair die is painted such that all 6 faces are blue. Each move, Daniel flips the coin and rolls the die. He then paints the face facing up on the die the color of the side facing up on the coin. The probability that the die is completely red after 7 moves is $\frac{p}{12^q}$, where p and q are positive integers such that p is not divisible by 12. What is $p + q$?

(A) 35 **(B)** 75 **(C)** 110 **(D)** 180 **(E)** 215

21. What is the sum of the digits of the remainder when

$$(4^2 - 9)(5^2 - 9)(6^2 - 9) \cdots (93^2 - 9)$$

is divided by 97?

(A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

22. A positive integer with $2n$ digits is *twisted* if the last n digits is some permutation of the first n digits, and the leading digit is nonzero. Let N be the number of twisted 6-digit integers. What is the sum of the digits of N?

(A) 18 **(B)** 21 **(C)** 24 **(D)** 27 **(E)** 30

23. Andy and Aidan want to meet up at school to exchange phone numbers. Before meeting, they each choose a random time between 12:00 PM and 1:30 PM to arrive. After arriving, Andy will wait 40 minutes before leaving, while Aidan will wait 10 minutes before leaving. Later, Andy learns that he has an unexpected recital he has to attend at 1:10 PM, so if Andy is waiting for Aidan to arrive and the time passes 1:10 PM, Andy will leave, and if Andy chose a time after 1:10 PM, Andy will not come to the meeting at all. What is the probability that Andy and Aidan meet?

(A) $\frac{2}{9}$ **(B)** $\frac{37}{162}$ **(C)** $\frac{53}{162}$ **(D)** $\frac{29}{81}$ **(E)** $\frac{7}{18}$

24. In acute $\triangle ABC$ with circumcircle Γ, the lines tangent to Γ at points B and C intersect at point D. Line segment $\overline{AD}$ intersects Γ at another point E, distinct from A, and $\overline{BC}$ at a point F. If $BF = 8$, $CF = 6$, and $EF = 4$, then the area of $\triangle BCD$ can be expressed as $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. What is $m + n$?

(A) 146 **(B)** 147 **(C)** 148 **(D)** 149 **(E)** 150

25. Consider the quadratic equation $P(x) = ax^2 + bx + 144$, where a and b are real numbers. It is known that $P(x)$ has two distinct positive integer roots, and its graph is tangent to the graph of $y = -x^2$. The sum of all possible values of $\frac{1}{a}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

(A) 61 **(B)** 62 **(C)** 63 **(D)** 64 **(E)** 65

SEASON 2 TMC 10

DO NOT OPEN UNTIL THURSDAY, December 31, 2020

Online **T**est **S**nowy **S**eries

Questions and comments about problems and solutions for this exam should be sent by PM to:

Emathmaster and **kevinmathz**.

The Season 2 OTIE will be held from January 8, 2021, to January 22, 2021. It is a 15-question, 3-hour, integer-answer exam. You will not be invited, but rather encouraged, to participate based on your score on this competition. *A complete listing of our publications may be found at our web site:*

`https://online-test-seasonal-series.github.io`

****Administration On An Earlier Date Will Literally Be Impossible****

1. All the information needed to administer this exam is contained in the non-existent TMC 10/12 Teacher's Manual.
2. YOU must not verify on the non-existent TMC 10/12 COMPETITION CERTIFICATION FORM that you followed all rules associated with the administration of the exam.
3. Send DeToasty3, Emathmaster, jeteagle, and kevinmathz a PM submitting your answers to the TMC 10. AoPS is strongly recommended and is the only way to submit your answers.
4. The publication, reproduction, or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, Discord, Facebook, Hangouts or other digital media of any type during this period is a violation of the competition rules.

The **Season 2 Online Snowy Tests** *was made possible by the contributions of the following people:*

Aathreyakadambi, AIME12345, ARMLlegend, Awesome_guy, AwesomeYRY, azduncan, DeToasty3, Emathmaster, Ish_Sahh, ivyzheng, jeteagle, kevinmathz, MathCounts145, NJOY, Orestis_Lignos, PCChess, reaganchoi, rqhu, VulcanForge, & zhao_andrew

Finally, we thank you for taking this mock. We hope you enjoyed it!