Season 2 TMC 10 Solutions

Online Test Snowy Series

December 31, 2020, to January 13, 2021

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Answer Key:

1. (D)	2. (C)	3. (A)	4. (C)	5. (E)
6. (E)	7. (E)	8. (C)	9. (B)	10. (A)
11. (C)	12. (B)	13. (A)	14. (D)	15. (A)
16. (B)	17. (B)	18. (D)	19. (C)	20. (E)
21. (D)	22. (A)	23. (C)	24. (E)	25. (A)

Solutions:

1. In square $ABCD$, let M be the midpoint of side $\overline{CD}$, and let N be the reflection of M over side $\overline{AB}$. What fraction of $\triangle MND$ lies within $ABCD$?

   (A) $\frac{1}{2}$ **(B)** $\frac{5}{8}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{3}{4}$ **(E)** $\frac{7}{8}$

   Proposed by PCChess

   Answer (D): Let segment $\overline{MN}$ intersect side $\overline{AB}$ at point X and let segment $\overline{ND}$ intersect side $\overline{AB}$ at point Y. We have that triangles NYX and NDM are similar with ratio $\frac{1}{2}$. Thus, the area of NYX is $\frac{1}{4}$ of the area of NDM. The remaining part of NDM (besides NYX) lies within the square while triangle NYX does not. Thus, we have that $\boxed{\textbf{(D)}\ \frac{3}{4}}$ of its area lies within square $ABCD$. ■

2. Jack is reading a 100 page book. He reads two pages every minute. After every 12 pages he reads, he takes a one minute break, and then he goes back to reading. If Jack starts reading at 2:00, what time will it be when he finishes reading his book?

(A) 2:32 **(B)** 2:56 **(C)** 2:58 **(D)** 3:05 **(E)** 3:18

Proposed by ivyzheng

Answer (C): For every 12 full pages, we see that $\frac{12}{2} + 1 = 7$ minutes pass. This will happen 8 times. The last $100 - 96 = 4$ pages will take $\frac{4}{2} = 2$ minutes to read. Therefore, it will take Jack a total of $7 \cdot 8 + 2 = 58$ minutes to finish the book. Adding this on to the time he starts reading, we get that he finishes reading at $\boxed{\textbf{(C)}\ 2{:}58}$. ■

3. Let a be the the largest solution to the equation

$$(x^2 + 6x + 8)(x^2 - 16x + 55) = 0,$$

and let b be the smallest solution. What is $a - b$?

(A) 15 **(B)** 16 **(C)** 17 **(D)** 18 **(E)** 22

Proposed by Emathmaster

Answer (A): The equation factors as $(x + 4)(x + 2)(x - 5)(x - 11) = 0$. Since the largest solution is $x = 11$ and the smallest is $x = -4$, our difference is $\boxed{\textbf{(A)}\ 15}$. ■

4. Pedro currently has 2 quarters, 3 dimes and 2 pennies. If he can only obtain quarters, dimes, nickels, and pennies, what is the minimum number of coins he needs to earn in order to reach a total of exactly 1 dollar?

(A) 2 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 10

Proposed by PCChess

Answer (C): The amount of money that Pedro currently has (in cents) is $2 \cdot 25 + 3 \cdot 10 + 2 \cdot 1 = 82$ cents. In order to reach exactly 1 dollar, he needs exactly 18 more cents. To achieve that with the minimum number of coins, we first need as many dimes as possible–there can only be at most 1 dime, which leaves us with 8 cents left. We can "fit" 1 nickel, and then 3 pennies. This gives us at least $1 + 1 + 3 = \boxed{\textbf{(C)}\ 5}$ coins. ■

5. 15 students are to be randomly split into 5 groups of 3 to work on a project. Alice, Bob, and Cooper are three of the students. Given that Alice and Bob are in the same group, what is the probability that Cooper is not in that group?

(A) $\frac{4}{7}$ **(B)** $\frac{3}{4}$ **(C)** $\frac{4}{5}$ **(D)** $\frac{8}{9}$ **(E)** $\frac{12}{13}$

Proposed by DeToasty3

Answer (E): If we fix Alice and Bob in the same group, there are 13 spots remaining, where one of them is in the same group as Alice and Bob. Thus, the probability that Cooper is not in that group is the complement, or $\boxed{\textbf{(E)}\ \frac{12}{13}}$. ■

6. Mark wants to distribute all 100 pieces of his candy to his five children, Albert, Bob, Charlie, Diana and Ethan. Diana and Ethan insist on each having a prime number of candies whose sum is also a prime number. Charlie insists on having exactly 35 candies, exactly 1 more than Albert and Bob's amounts combined. Given that Ethan has the smallest number of candies, how many candies must Mark give to Diana?

   (A) 3 **(B)** 7 **(C)** 17 **(D)** 23 **(E)** 29

   Proposed by ivyzheng

   Answer (E): If Diana and Ethan have a prime number of candies with the sum also being a prime number, then either Diana or Ethan must have 2 candies. Since it is given that Ethan has the smallest number of candies, we find that Ethan must have 2 candies. If Charlie has 1 more candy that Albert and Bob's amounts combined, then the sum of Albert and Bob's candies is 34. Since the total number of candies is 100, and the sum of Albert, Bob, Charlie, and Ethan's amounts is $34 + 35 + 2 = 81$, we have that the number of candies Mark gives to Diana is $\boxed{\textbf{(E)}\ 29}$, which we know is a prime number. ■

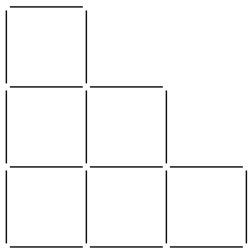
7. When all 9 diagonals of a regular hexagon are drawn, they partition the hexagon into some number of individual regions. What percent of these regions are triangular?

   (A) 25% **(B)** 37.5% **(C)** 50% **(D)** 62.5% **(E)** 75%

   Proposed by Emathmaster

   Answer (E): Drawing the hexagon and all of its 9 diagonals, we find that the hexagon is split into 24 regions. Of these, we can count that there are exactly 18 triangular regions. This gives $\frac{18}{24} = \boxed{\textbf{(E)}\ 75\%}$ of the regions are triangular. ■

8. A 3-step staircase is shown below, where each side of each square is of unit length.



Extend this pattern to create two 2021-step staircases. When these two staircases are fit together to form a 2021×2022 rectangle, the two staircases meet each other at a crease of length L. What is L? (In the resulting rectangle, a crease is defined as the total length of the segments touched by both of the original staircases.)

(A) 2022 **(B)** 2023 **(C)** 4041 **(D)** 4042 **(E)** 4043

Proposed by Emathmaster

Answer (C): We experiment with small staircase-lengths and attempt to find a pattern. When the staircase has 2 steps, the crease has length $2 \cdot 2 - 1 = 3$. When the staircase has 3 steps, the crease has length $2 \cdot 3 - 1 = 5$. In general, the crease for 2 staircases of length n is $2n - 1$. Here, $n = 2021$ and thus the crease has length $2 \cdot 2021 - 1 = \boxed{\textbf{(C) } 4041}$. ■

9. Let the area of equilateral $\triangle ABC$ be 9. Let O denote the center of its circumcircle. Let ω_1 and ω_2 be circles centered at A and B, respectively, such that they both pass through O. Let ω_1 and ω_2 intersect at P, distinct from O. What is the area of $CBPA$?

   (A) $6\sqrt{3}$ **(B)** 12 **(C)** $8\sqrt{3}$ **(D)** $9\sqrt{3}$ **(E)** 15

   Proposed by PCChess

   Answer (B): We claim that triangles APB and AOB are congruent. They share side AB, and $PA = PB = OA = OB$ since they are all radii of the two congruent circles, and thus they are congruent by SSS congruence, so they have the same area. So, $[CBPA] = [ABC] + [APB] = [ABC] + [AOB] = [ABC] + \frac{1}{3}[ABC] = \frac{4}{3}[ABC] = \frac{4}{3} \cdot 9 = \boxed{\textbf{(B) } 12}$. (Note that $[ABC]$ means the area of triangle ABC, etc.) ■

10. In a regular hexagon with side length 2, three of the sides are chosen at random. Next, the midpoints of each of the chosen sides are drawn. What is the probability that the triangle formed by the three midpoints has a perimeter which is an integer?

(A) $\frac{1}{10}$ **(B)** $\frac{1}{5}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{2}{5}$ **(E)** $\frac{1}{2}$

Proposed by DeToasty3

Answer (A): We see that there are three possible lengths of a side of the triangle: connecting two adjacent sides, connecting two sides one side apart, and connecting two opposite sides.

For the adjacent sides, we see that the length is one-half the length of the segment connecting two vertices with one vertex in between, which is $\frac{1}{2} \cdot 2\sqrt{3} = \sqrt{3}$, which is not an integer. For the sides one side apart, we see that the length is one-half the average of the side length and the length of the segment connecting two vertices with two vertices in between, which is $\frac{1}{2} \cdot (2+4) = 3$. Finally, for the two opposite sides, we see that the length is equal to the length of the segment connecting two vertices with one vertex in between, which is $2\sqrt{3}$.

Of these, we see that only the middle case has an integer length. There are two possible triangles with all three sides of this length. In total, there are $\binom{6}{3} = 20$ possible triangles, so our final probability is $\dfrac{2}{20} = \boxed{\textbf{(A) } \dfrac{1}{10}}$. ■

11. A positive integer is called a *flake* if it has at least three distinct prime factors. Two flakes are defined to be in a *snowflake* if there exists a prime that is divisible by the greatest common divisor of the two flakes. When two flakes in a snowflake are multiplied, what is the smallest possible number of divisors in the resulting number?

(A) 18 **(B)** 27 **(C)** 48 **(D)** 54 **(E)** 64

Proposed by Emathmaster

Answer (C): Let the first flake be $p_1^{e1}p_2^{e2}p_3^{e3}$ (note that it is optimal to have less prime factors). Since the greatest common divisor (GCD) of the two flakes divides a prime, it is either a prime or 1. If the GCD is 1, then the other flake has 3 other distinct prime factors, all different from the prime factors of the original flake. This gives $(1+1)^6 = 64$ divisors of the product. If the GCD is a prime, then the two flakes share 1 prime factor. Thus, the number of divisors of the product is $(2+1)(1+1)^4 = 48$. The smallest possible number of divisors is $\boxed{\textbf{(C) } 48}$. ■

12. A group of people are in a room. It is given that 5 people have a pet dog, 6 people have a pet cat, 8 people have a pet fish, and 3 people have no pets. If no one has more than two pets, and no one has more than one of the same type of pet, what is the smallest possible number of people in the room?

(A) 10 **(B)** 13 **(C)** 14 **(D)** 16 **(E)** 19

Proposed by DeToasty3

Answer (B): To minimize the number of people in the room, we want to minimize the number of people who have at least one pet. In total, there are $5 + 6 + 8 = 19$ pets in the room. Thus, our minimum should be $\frac{19}{2} + \frac{1}{2} = 10$ people with at least one pet. It suffices to find a construction. Let there be x people with both a pet dog and a pet cat, y people with both a pet cat and a pet fish, and z people with both a pet dog and a pet fish. Note that one person should have only a pet fish, so our 7 comes from $8 - 1$. Then, we have the following system of equations:

$$\begin{aligned} x + y &= 6, \\ y + z &= 7, \\ x + z &= 5. \end{aligned}$$

From this, we get $x = 2$, $y = 4$, and $z = 3$. This shows that the smallest number of people in the room is $10 + 3 = \boxed{\textbf{(B)}\ 13}$, accounting for the three people who have no pets. ■

13. A function f is defined by a real-valued expression

$$f(x) = \frac{2020x + 1}{x + 2020} + \frac{\sqrt[4]{|x| - 2021} - \sqrt[8]{2021 - |x|}}{|x - 2021|}.$$

For x in the domain of f, $f(x)$ can be simplified to a value N. What is the remainder when N is divided by 100?

(A) 19 **(B)** 21 **(C)** 41 **(D)** 59 **(E)** N is not an integer.

Proposed by NJOY

Answer (A): Note that we require $|x| - 2021 = 2021 - |x| = 0$ due to domain balancing. Then, due to the denominator being $\neq 0$, we get that $x = -2021$. So, $N = 2020 \cdot 2021 - 1 \equiv \boxed{\textbf{(A)}\ 19} \pmod{100}$. ■

14. For a positive integer n, define a function $f(n)$ to be equal to the largest integer k such that n is divisible by 2^k. For example, $f(8) = 3$ and $f(12) = 2$. Now, let p, q, and r be distinct primes less than 100, so that M is the largest value that

$$f((p + 1)(q + 1)(r + 1))$$

can take, and m is the smallest value. What is $M + m$?

(A) 11 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16

Proposed by Emathmaster

Answer (D): Note that if we want to maximize $f((p+1)(q+1)(r+1))$, we will have to make p, q, r be one less than a number with a large exponent for 2. Since p, q, r are less than 100, the best that we can do is 64, a multiple of 32, a multiple of 16, and so on. Since $64 - 1 = 63$, which is not prime, we move on to check multiples of 32. We see that $32 - 1 = 31$, which is a prime, but $96 - 1 = 95$, which is not a prime (We have already checked 64, so we may skip it.). Next, we see that $48 - 1 = 47$, which is a prime, and $80 - 1 = 79$, which is a prime. We see that this is the best that we can do, where 32, 48, and 80 contribute 5, 4, and 4 to our count, respectively, giving $M = 5 + 4 + 4 = 13$.

Next, to minimize $f((p+1)(q+1)(r+1))$, we can let one of p, q, r be 2 and the other two be primes which are 1 modulo 4, e.g. 5 and 13. We see that this is the best that we can do, where 3, 6, and 14 contribute 0, 1, and 1 to our count, respectively, giving $m = 0 + 1 + 1 = 2$.

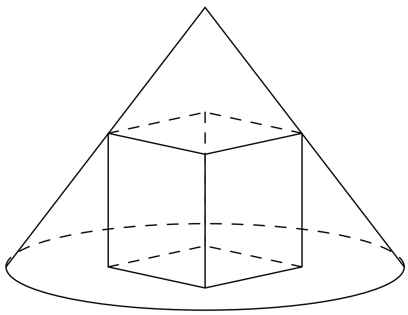
Thus, our answer is $M + m = 13 + 2 = \boxed{\textbf{(D) } 15}$. ■

15. A right circular cone has base radius 3 and height 4. Let s be the side length of the largest possible cube that can be inscribed inside the cone such that two of its faces are parallel to the base of the cone. Then $s = a - b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by the square of any prime. What is $a + b + c$?

(A) 62 **(B)** 64 **(C)** 66 **(D)** 68 **(E)** 70

Proposed by Awesome_guy

Answer (A):



Using the diagram, we can set up an equation:

$$\frac{8-2s}{s\sqrt{2}} = \frac{4}{3}.$$

This equation reduces to

$$24 - 6s = 4s\sqrt{2} \implies s(4\sqrt{2}+6) = 24 \implies s(2\sqrt{2}+3) = 12$$

$$\implies s = \frac{12(2\sqrt{2}-3)}{(2\sqrt{2}+3)(2\sqrt{2}-3)} \implies s = 36 - 24\sqrt{2}.$$

From here, we get that our answer is $36 + 24 + 2 = \boxed{\textbf{(A) } 62}$. ■

16. Bob and Bill are playing with 7 red cards and 3 blue cards. The cards are numbered from 1 to 10, inclusive, and are flipped over so that neither person can see the numbers. Bill knows that the blue cards are numbered 1, 8, and 9, in some order, but Bob does not know this, nor does he know that Bill knows. Bill chooses a card first, and then Bob chooses a different card. A player wins the game if they choose a card with a larger number. If both players play optimally, what is the probability that Bill wins?

(A) $\frac{1}{2}$ **(B)** $\frac{5}{9}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{7}{10}$ **(E)** $\frac{25}{27}$

Proposed by ivyzheng

Answer (B): The expected value of a red card is $\frac{2+3+4+5+6+7+10}{7} = \frac{37}{7}$, and the expected value of a blue card is $\frac{1+8+9}{3} = 6$. We have that $6 > \frac{37}{7}$, so it is optimal for Bill to choose a blue card.

If Bill chooses a 1, then Bob's number will always be greater. If Bill chooses an 8, then Bob's number will be smaller with a $\frac{7}{9}$ probability. If Bill chooses a 9, then Bob's number will be smaller with a $\frac{8}{9}$ probability.

Therefore, the final probability is $\frac{1}{3} \cdot 0 + \frac{1}{3} \cdot \frac{7}{9} + \frac{1}{3} \cdot \frac{8}{9} = \boxed{\textbf{(B) } \frac{5}{9}}$. ■

17. Farmers A and B are mowing a rectangular grid of 21 rows and 23 columns of cells of grass. A starts from the top-left cell of the grid and mows right until he has mowed all the grass in the row. Then, B starts from where A stopped and mows down the same column until he has mowed all the grass in the column. They keep taking turns mowing, each time turning 90° clockwise, switching the farmer, and mowing only unmowed grass in the same direction until not possible. If A and B mow at rates of 7 and 3 cells per minute, respectively, and it takes no time to turn 90° clockwise and switch farmers, how long, in minutes, will it take them to finish mowing all the grass?

(A) 108 **(B)** 109 **(C)** 110 **(D)** 111 **(E)** 112

Proposed by Emathmaster

Answer (B): Observe that Farmer A mows horizontally over $23 + 22 + 21 + \cdots + m$ cells, and Farmer B mows vertically over $20 + 19 + 18 + \cdots + 2 + 1$ cells. Note that the m actually ends at 3 instead of 1 or 2 because $23 - 21 = 2$, so Farmer A will not mow a segment of length 1 or 2. Therefore, the total time spent is equal to

$$\frac{23 + 22 + 21 + \cdots + 4 + 3}{7} + \frac{20 + 19 + 18 + \cdots + 2 + 1}{3},$$

which upon simplification equals $\boxed{\textbf{(B)}\ 109}$ minutes. ■

18. A sequence is defined recursively by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all integers $n \geq 2$. Let S be the sum of all positive integers n such that $|F_n - n^2| \leq 3n$. What is the sum of the digits of S?

(A) 6 **(B)** 7 **(C)** 10 **(D)** 11 **(E)** 14

Proposed by Emathmaster

Answer (D): We list out the first 14 values of F_n and n^2 (for $n = 1$ through $n = 14$). We find that values $1, 2, 3$ work. Then, 11 and 12 also work. After that, the Fibonacci numbers grow way faster than the perfect squares and thus there are no more solutions. Thus, $S = 1 + 2 + 3 + 11 + 12 = 29$ and thus the sum of its digits is $\boxed{\textbf{(D)}\ 11}$. ■

19. In triangle ABC, $AB = 15$ and $BC = 20$, with a right angle at B. Point D is chosen on side $\overline{AC}$ and is reflected over sides $\overline{AB}$ and $\overline{BC}$ (not consecutively) to create points M and N, respectively. What is the smallest possible value of the length MN?

(A) 18 **(B)** 21 **(C)** 24 **(D)** 27 **(E)** 30

Proposed by ARMLlegend

Answer (C): Let P be the intersection of $\overline{DN}$ and $\overline{BC}$, and let Q be the intersection of $\overline{DM}$ and $\overline{AB}$. We note that $\triangle ABC$ is a right triangle with right angle at B. Thus, $PBQD$ is a rectangle. We also see that $\triangle DMN$ is a dilation of $\triangle DQP$ with scale factor 2 and dilation center D. Since $PBQD$ is a rectangle, we have that $BD = PQ$. Thus, we see that $MN = 2PQ = 2BD$. Thus, we need to minimize BD. We see that BD is minimized when $\overline{BD}$ is perpendicular to $\overline{AC}$, and here, we find that $BD = 12$, so the smallest possible value of MN is $\boxed{\textbf{(C)}\ 24}$. ■

20. A fair coin is painted such that one side is red and the other side is blue. A fair die is painted such that all 6 faces are blue. Each move, Daniel flips the coin and rolls the die. He then paints the face facing up on the die the color of the side facing up on the coin. The probability that the die is completely red after 7 moves is $\frac{p}{12^q}$, where p and q are positive integers such that p is not divisible by 12. What is $p+q$?

    (A) 35 **(B)** 75 **(C)** 110 **(D)** 180 **(E)** 215

    Proposed by ivyzheng

    Answer (E): We have that at most 1 move can be a repeat or a blue side painted blue. If a blue side is painted blue, we have 21 ways to choose the roll where this occurs. The other rolls are $\frac{5!}{6^5}$ chances and we multiply by $\frac{21}{6}$ to get $\frac{2520}{6^6}$ chance. Each occasion must follow a RB-pattern so we divide by 2^7. Now if we have seven red rolls, we have one space that's a repeat so we have $3 \cdot 7!$ ways to organize that with denominator $6^7 \cdot 2^7$. That gives $\frac{2520}{6^6}$ too, so we have $\frac{7!}{6^6 \cdot 2^7} = \frac{840}{6^5 \cdot 2^7} = \frac{210}{12^5}$. Our desired answer is $210 + 5 = \boxed{\textbf{(E) } 215}$. ■

21. What is the sum of the digits of the remainder when

    $$(4^2-9)(5^2-9)(6^2-9)\cdots(93^2-9)$$

    is divided by 97?

    (A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

    Proposed by Ish_Sahh

    Answer (D): Since $n^2 - 9 = (n-3)(n+3)$, we have that

    $$\begin{aligned} &(4^2-9)(5^2-9)(6^2-9)\cdots(93^2-9) \\ &= \frac{96! \cdot 90!}{6!} \\ &= \frac{96! \cdot 96!}{6! \cdot 96 \cdot 95 \cdot 94 \cdot 93 \cdot 92 \cdot 91}. \end{aligned}$$

    Since 97 is prime, using (mod 97) and Wilson's Theorem, we get that

    $$\begin{aligned} &\frac{96! \cdot 96!}{6! \cdot 96 \cdot 95 \cdot 94 \cdot 93 \cdot 92 \cdot 91} \\ &\equiv \frac{1}{6! \cdot (-6)(-5)(-4)(-3)(-2)(-1)} \pmod{97} \\ &\equiv \frac{1}{(6!)^2} \pmod{97} \\ &\equiv \frac{1}{720^2} \pmod{97}. \end{aligned}$$

To solve this, we can use $720^2 r \equiv 1 \pmod{97}$, where r is our desired remainder. Since 776 is a multiple of 97, 720 can be written as $(776 - 56)$ which makes the congruence be $(776-56)^2 r \equiv 1 \pmod{97}$ which would give $56^2 r \equiv 1 \pmod{97}$. 56^2 is 3136, and 3104 is a multiple of 97, so the congruence is $32r \equiv 1 \pmod{97}$. Since $32 \cdot -3 = -96$ and $-96 \equiv 1 \pmod{97}$, $r = -3 + 97 = 94$ and the sum of the digits of 94 is $\boxed{\textbf{(D)}\ 13}$.
■

22. A positive integer with $2n$ digits is *twisted* if the last n digits is some permutation of the first n digits, and the leading digit is nonzero. Let N be the number of twisted 6-digit integers. What is the sum of the digits of N?

(A) 18 **(B)** 21 **(C)** 24 **(D)** 27 **(E)** 30

Proposed by PCChess

Answer (A): We do casework based on the number of repeating digits in the first n digits.

Case 1. All 3 digits are the same. Then there are 9 ways to choose the first 3 digits. There are 9 numbers here.

Case 2. 2 digits are the same. We do casework again on if there is a zero.

- **Subcase 1.** If there is one 0, there are 9 ways to choose the other digit, and 2 ways to place the 0. Further, there are 3 ways to arrange the same digits in the last n place values, so there are $9 \cdot 2 \cdot 3 = 54$ numbers here.
- **Subcase 2.** If there are 2 zeros, there are 9 ways to choose the other digit, and 3 ways to arrange the digits in the second half. Our total here is $9 \cdot 3 = 27$.
- **Subcase 3.** If there are no zeros, there are 9 ways to choose the digit we use twice, 8 ways to choose the digit we use once, 3 ways to arrange the digits in the first n place values, and 3 ways to arrange the digits in the last n place values. Our total here is $9 \cdot 8 \cdot 3 \cdot 3 = 648$.

Our total here is $54 + 27 + 648 = 729$.

Case 3. All 3 digits are different. We do casework on if there is a zero.

- **Subcase 1.** If there is one zero, there are 2 ways to place the 0 and $9 \cdot 8$ ways to choose the other 2 digits. There are also 6 ways to arrange the digits in the latter half, so our total here is $9 \cdot 8 \cdot 2 \cdot 6 = 864$.
- **Subcase 2.** If there are no zeros, there are $9 \cdot 8 \cdot 7$ ways to choose the first n digits, and 6 ways to arrange them in the latter half. Our total here is $9 \cdot 8 \cdot 7 \cdot 6 = 3024$.

Our total here is $864 + 3024 = 3888$.

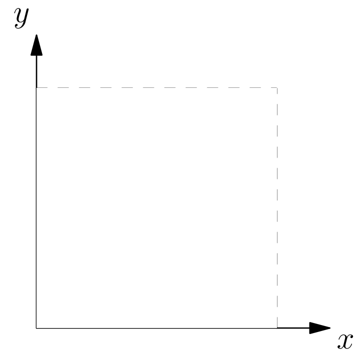
Summing, we get that $N = 9 + 729 + 3888 = 4626$, so our answer is $\boxed{\textbf{(A)}\ 18}$. ■

23. Andy and Aidan want to meet up at school to exchange phone numbers. Before meeting, they each choose a random time between 12:00 PM and 1:30 PM to arrive. After arriving, Andy will wait 40 minutes before leaving, while Aidan will wait 10 minutes before leaving. Later, Andy learns that he has an unexpected recital he has to attend at 1:10 PM, so if Andy is waiting for Aidan to arrive and the time passes 1:10 PM, Andy will leave, and if Andy chose a time after 1:10 PM, Andy will not come to the meeting at all. What is the probability that Andy and Aidan meet?

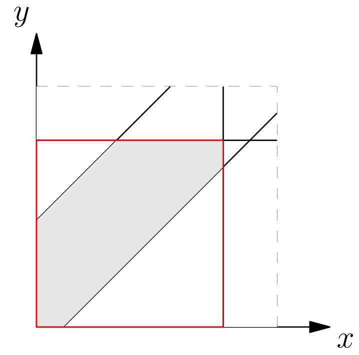
(A) $\frac{2}{9}$ **(B)** $\frac{37}{162}$ **(C)** $\frac{53}{162}$ **(D)** $\frac{29}{81}$ **(E)** $\frac{7}{18}$

Proposed by rqhu

Answer (C): Let the number of minutes after 12:00 PM that Andy arrives be x, and let the number of minutes after 12:00 PM that Aidan arrives be y. We set up a graph of what times are possible. We start with the axes and mark 0 and 90 and both axes (which represent the earliest and latest times they could arrive).



Suppose that Andy arrives before Aidan ($x < y$). Then Aidan must arrive in the next 40 minutes after Andy arrives, so $y - x \leq 40$. Similarly, if Aidan arrives before Andy, then Andy must arrive within 10 minutes of Aidan arriving, so $x - y \leq 10$. Finally, due to the unexpected meeting, the time that they meet must be before 1:10 PM, so $x, y < 70$. We add these inequalities to our diagram and mark the desired region:



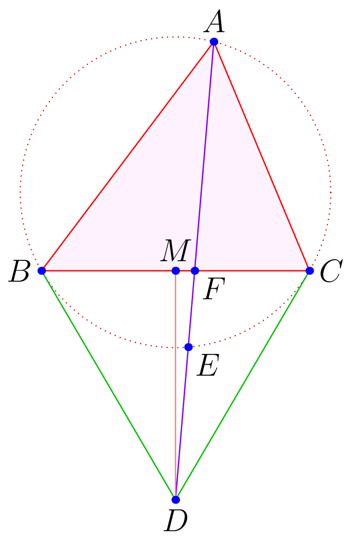
We find the area of this region by taking the area of the square outlined in red and then subtracting the areas of the right triangles in the upper left and bottom right. The area of the red square is $70^2 = 4900$. Since the slopes of the hypotenuses of the upper left and bottom right triangles are both 1, the triangles are isosceles right and the legs are equal. Note that the upper left triangle has its right angle vertex at $(0, 70)$ and one endpoint of the hypotenuse at $(0, 40)$. Thus, its leg length is $70 - 40 = 30$. Similarly, for the other triangle, one endpoint of the hypotenuse is at $(10, 0)$ and the right angle vertex is at $(70, 0)$. Thus, its leg length is $70 - 10 = 60$. Thus, the areas of the two triangles are $\frac{30^2}{2} = 450$ and $\frac{60^2}{2} = 1800$ respectively. The desired area is thus $4900 - 450 - 1800 = 2650$. The entire square's area is $90^2 = 8100$, so the desired probability is $\dfrac{2650}{8100} = \boxed{\textbf{(C) } \dfrac{53}{162}}$. ■

24. In acute $\triangle ABC$ with circumcircle Γ, the lines tangent to Γ at points B and C intersect at point D. Line segment $\overline{AD}$ intersects Γ at another point E, distinct from A, and $\overline{BC}$ at a point F. If $BF = 8$, $CF = 6$, and $EF = 4$, then the area of $\triangle BCD$ can be expressed as $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. What is $m + n$?

(A) 146 **(B)** 147 **(C)** 148 **(D)** 149 **(E)** 150

Proposed by Awesome_guy

Answer (E):



By Power of a Point, we have that $8 \cdot 6 = AF \cdot 4$, so $AF = 12$. Now, we wish to find DE. Note that $\triangle BCD$ is isosceles, so the foot of the perpendicular from D onto $\overline{BC}$

is its midpoint. Let M be the midpoint of $\overline{BC}$. Then, we have

$$BD^2 = DE(DE + 12 + 4)$$

by Power of a Point. Also, by the Pythagorean Theorem,

$$BD^2 = DM^2 + 7^2 = (DF^2 - FM^2) + 7^2.$$

Now, we see that $FM = BF - BM = 1$ and $DF = DE + 4$, so we obtain $DE = 8$, $DF = 12$, and $DM = \sqrt{143}$ after some simplifications. Finally, since $BC = 14$, we obtain that the area of $\triangle BCD$ is $\frac{1}{2} \cdot 14 \cdot \sqrt{143} = 7\sqrt{143}$. Thus, our final answer is $7 + 143 = \boxed{\textbf{(E) } 150}$. ■

25. Consider the quadratic equation $P(x) = ax^2 + bx + 144$, where a and b are real numbers. It is known that $P(x)$ has two distinct positive integer roots, and its graph is tangent to the graph of $y = -x^2$. The sum of all possible values of $\frac{1}{a}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

(A) 61 **(B)** 62 **(C)** 63 **(D)** 64 **(E)** 65

Proposed by reaganchoi

Answer (A): The key is to reverse the quadratic into the form $144x^2+bx+a$ (although other methods probably work too). This must have two unit-fraction roots, $\frac{1}{r}$ and $\frac{1}{s}$. In addition, since the quadratic $(a+1)x^2+bx+144$ has exactly one real root, so must $144x^2+bx+(a+1)$; in other words, our quadratic of $144x^2+bx+a$ has a vertex with y-coordinate -1.

The roots are $\frac{1}{r}$ and $\frac{1}{s}$; WLOG, let $r > s$ so $\frac{1}{s} > \frac{1}{r}$. Then, the vertex must be at $\left(\frac{\frac{1}{r}+\frac{1}{s}}{2}, -1\right)$. Writing this in vertex form:

$$144\left(x - \frac{\frac{1}{r}+\frac{1}{s}}{2}\right)^2 - 1$$

Note that plugging in $x = \frac{1}{r}$ gives 0. Thus, we have

$$144\left(\frac{\frac{1}{s}-\frac{1}{r}}{2}\right)^2 - 1 = 0.$$

$$36\left(\frac{1}{s} - \frac{1}{r}\right)^2 = 1.$$

$$\frac{1}{s} - \frac{1}{r} = \frac{1}{6}.$$

Then, we can expand this to get $6r - 6s = rs$, so $rs - 6r + 6s = 0$. Solving this Diophantine equation gives $(r+6)(s-6) = -36$; since $r, s > 0$, we need $r + 6 > 6$, so our only possible product pairs are $(9, -4)$, $(12, -3)$, $(18, -2)$, and $(36, -1)$, giving pairs $(r, s) = (3, 2)$, $(6, 3)$, $(12, 4)$, and $(30, 5)$.

Note that, by Vieta's, $\frac{144}{a} = rs$. Thus, to find the sum of all $\frac{1}{a}$, we need to find the sum of all $\frac{rs}{144}$. This sum can be evaluated to be $\frac{6+18+48+150}{144} = \frac{222}{144} = \frac{37}{24}$, so the answer is $37 + 24 = \boxed{\textbf{(A) } 61}$. ■