OTSS TMC

*Snowy Mathematics Competitions*

Snowy Mathematics Competitions

2nd Annual

# TMC 12

Thursday, December 31, 2020

## INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU TELL YOURSELF.
2. This is a twenty-five question multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. Mark your answer to each problem on a TMC 12 Private Message with a keyboard. Check the keys for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded; however, this mock will be graded by people.
4. SCORING: You will receive **6** points for each correct answer, **1.5** points for each problem left unanswered, and **0** points for each incorrect answer.
5. No aids are permitted other than scratch paper, graph paper, rulers, compass, protractors, and erasers. No calculators, smartwatches, or computing devices are allowed. No problems on the test will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your proctor will not ask you to record certain information on the private message.
8. When you give the signal, begin working on the problems. You will have 75 minutes to complete the test. You can discuss only with people that have taken the test during the period when make-ups are eligible.
9. When you finish the exam, don't sign your name in the space provided on the Private Message.

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The Committee on the Test Mathematics Competitions reserves the right to re-examine students before deciding whether to grant official status to their scores. The Committee also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

*Students who score well on this TMC 12 will not be invited, but rather encouraged, to take the Season 2 Online Test Invitational Examination (OTIE) from January 8, 2021, to January 22, 2021. More details about the OTIE and other information are not on the back page of this test booklet.*

1. In square $ABCD$, let $M$ be the midpoint of side $\overline{CD}$, and let $N$ be the reflection of $M$ over side $\overline{AB}$. What fraction of $\triangle MND$ lies within $ABCD$?

   **(A)** $\frac{1}{2}$ **(B)** $\frac{5}{8}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{3}{4}$ **(E)** $\frac{7}{8}$

2. Jack is reading a 100 page book. He reads two pages every minute. After every 12 pages he reads, he takes a one minute break, and then he goes back to reading. If Jack starts reading at 2:00, what time will it be when he finishes reading his book?

   **(A)** 2:32 **(B)** 2:56 **(C)** 2:58 **(D)** 3:05 **(E)** 3:18

3. Pedro currently has 2 quarters, 3 dimes and 2 pennies. If he can only obtain quarters, dimes, nickels, and pennies, what is the minimum number of coins he needs to earn in order to reach a total of exactly 1 dollar?

   **(A)** 2 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 10

4. For how many integer values of $b$ does the equation $2x^2 + bx + 5 = 0$ not have any real solutions?

   **(A)** 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

5. Mark wants to distribute all 100 pieces of his candy to his five children, Albert, Bob, Charlie, Diana and Ethan. Diana and Ethan insist on each having a prime number of candies whose sum is also a prime number. Charlie insists on having exactly 35 candies, exactly 1 more than Albert and Bob's amounts combined. Given that Ethan has the smallest number of candies, how many candies must Mark give to Diana?

   **(A)** 3 **(B)** 7 **(C)** 17 **(D)** 23 **(E)** 29

6. Positive real numbers $a$, $b$, $c$, $d$, and $e$ satisfy $a+b+c+d+e = 2020$. Let $S$ denote the sum of the minimum and maximum values of $\lfloor a \rfloor + \lfloor b \rfloor + \lfloor c \rfloor + \lfloor d \rfloor + \lfloor e \rfloor$, where $\lfloor r \rfloor$ denotes the largest integer less than or equal to $r$ for all real numbers $r$. What is the sum of the digits of $S$?

   **(A)** 12 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16

7. In a regular hexagon with side length 2, three of the sides are chosen at random. Next, the midpoints of each of the chosen sides are drawn. What is the probability that the triangle formed by the three midpoints has a perimeter which is an integer?

**(A)** $\frac{1}{10}$ **(B)** $\frac{1}{5}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{2}{5}$ **(E)** $\frac{1}{2}$

8. A positive integer is called a *flake* if it has at least three distinct prime factors. Two flakes are defined to be in a *snowflake* if there exists a prime that is divisible by the greatest common divisor of the two flakes. When two flakes in a snowflake are multiplied, what is the smallest possible number of divisors in the resulting number?

**(A)** 18 **(B)** 27 **(C)** 48 **(D)** 54 **(E)** 64

9. A group of people are in a room. It is given that 5 people have a pet dog, 6 people have a pet cat, 8 people have a pet fish, and 3 people have no pets. If no one has more than two pets, and no one has more than one of the same type of pet, what is the smallest possible number of people in the room?

**(A)** 10 **(B)** 13 **(C)** 14 **(D)** 16 **(E)** 19

10. Let $A$ and $B$ be two distinct points in the plane. A ray is drawn emanating from $A$ in any random direction, and a ray is drawn emanating from $B$ in any random direction. What is the probability that the two rays intersect?

**(A)** $\frac{1}{8}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{3}{8}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{3}{4}$

11. For a positive integer $n$, define a function $f(n)$ to be equal to the largest integer $k$ such that $n$ is divisible by $2^k$. For example, $f(8) = 3$ and $f(12) = 2$. Now, let $p$, $q$, and $r$ be distinct primes less than 100, so that $M$ is the largest value that

$$f((p+1)(q+1)(r+1))$$

can take, and $m$ is the smallest value. What is $M + m$?

**(A)** 11 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16

12. A solution to $z^{27} - 1 = 0$ is chosen at random. What is the probability that it is also a solution to $\omega^6 + \omega^3 + 1 = 0$?

**(A)** $\frac{1}{27}$ **(B)** $\frac{1}{9}$ **(C)** $\frac{4}{27}$ **(D)** $\frac{5}{27}$ **(E)** $\frac{2}{9}$

13. A sequence is defined recursively by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all integers $n \geq 2$. Let $S$ be the sum of all positive integers $n$ such that $|F_n - n^2| \leq 3n$. What is the sum of the digits of $S$?

    **(A)** 6 **(B)** 7 **(C)** 10 **(D)** 11 **(E)** 14

14. Let $ABC$ be a triangle with circumcenter $O$ and side lengths $AB = 9$, $BC = 7$, and $AC = 8$. If $D$ is the reflection of $A$ across line $BO$, then the length $AD$ can be written as $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. What is $m + n$?

    **(A)** 25 **(B)** 43 **(C)** 44 **(D)** 50 **(E)** 57

15. For how many positive integers $n \leq 15$ does there exist a positive integer $k$ such that

    $$\lfloor \log_2 k \rfloor + \lfloor \log_3 k \rfloor + \lfloor \log_4 k \rfloor + \cdots + \lfloor \log_8 k \rfloor = n?$$

    (Here, $\lfloor r \rfloor$ denotes the largest integer less than or equal to $r$ for all real numbers $r$.)

    **(A)** 8 **(B)** 9 **(C)** 12 **(D)** 13 **(E)** 15

16. A fair coin is painted such that one side is red and the other side is blue. A fair die is painted such that all 6 faces are blue. Each move, Daniel flips the coin and rolls the die. He then paints the face facing up on the die the color of the side facing up on the coin. The probability that the die is completely red after 7 moves is $\frac{p}{12^q}$, where $p$ and $q$ are positive integers such that $p$ is not divisible by 12. What is $p + q$?

    **(A)** 35 **(B)** 75 **(C)** 110 **(D)** 180 **(E)** 215

17. How many distinct cubic polynomials $P(x)$ with all integer coefficients and leading coefficient 1 exist such that $P(0) = 3$, $|P(1)| < 12$, and $P(x)$ has three (not necessarily real or distinct) roots whose squares sum to 34?

    **(A)** 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

18. A positive integer is called *edgy* if the sum of its digits divides the sum of the squares of its digits. For example, the number 19 is not an edgy number because $1 + 9 = 10$ does not divide $1^2 + 9^2 = 82$. How many two-digit edgy numbers are there?

    **(A)** 9 **(B)** 18 **(C)** 20 **(D)** 22 **(E)** 23

19. A positive integer with $2n$ digits is *twisted* if the last $n$ digits is some permutation of the first $n$ digits, and the leading digit is nonzero. Let $N$ be the number of twisted 6-digit integers. What is the sum of the digits of $N$?

**(A)** 18 **(B)** 21 **(C)** 24 **(D)** 27 **(E)** 30

20. Andy and Aidan want to meet up at school to exchange phone numbers. Before meeting, they each choose a random time between 12:00 PM and 1:30 PM to arrive. After arriving, Andy will wait 40 minutes before leaving, while Aidan will wait 10 minutes before leaving. Later, Andy learns that he has an unexpected recital he has to attend at 1:10 PM, so if Andy is waiting for Aidan to arrive and the time passes 1:10 PM, Andy will leave, and if Andy chose a time after 1:10 PM, Andy will not come to the meeting at all. What is the probability that Andy and Aidan meet?

**(A)** $\frac{2}{9}$ **(B)** $\frac{37}{162}$ **(C)** $\frac{53}{162}$ **(D)** $\frac{29}{81}$ **(E)** $\frac{7}{18}$

21. In acute $\triangle ABC$ with circumcircle $\Gamma$, the lines tangent to $\Gamma$ at points $B$ and $C$ intersect at point $D$. Line segment $\overline{AD}$ intersects $\Gamma$ at another point $E$, distinct from $A$, and $\overline{BC}$ at a point $F$. If $BF = 8$, $CF = 6$, and $EF = 4$, then the area of $\triangle BCD$ can be expressed as $m\sqrt{n}$, where $m$ and $n$ are positive integers, and $n$ is not divisible by the square of any prime. What is $m + n$?

**(A)** 146 **(B)** 147 **(C)** 148 **(D)** 149 **(E)** 150

22. For how many ordered triples of integers $(a, b, c)$ between 1 and 10, inclusive, does

$$\frac{ab^2 + bc^2 + ca^2}{abc}$$

have a terminating (non-repeating) decimal expansion?

**(A)** 226 **(B)** 232 **(C)** 240 **(D)** 244 **(E)** 250

23. Consider the quadratic equation $P(x) = ax^2 + bx + 144$, where $a$ and $b$ are real numbers. It is known that $P(x)$ has two distinct positive integer roots, and its graph is tangent to the graph of $y = -x^2$. The sum of all possible values of $\frac{1}{a}$ can be written as $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. What is $m + n$?

**(A)** 61 **(B)** 62 **(C)** 63 **(D)** 64 **(E)** 65

24. Let $\triangle ABC$ have $\angle BAC = 60^\circ$ and incenter $I$. Let $\omega_A$ be a circle in the exterior of $\triangle ABC$ that is tangent to side $\overline{BC}$ and the extensions of the other two sides. Let $I_A$ be the center of $\omega_A$. If $BI_A = 4$ and $CI_A = 3$, then the area of quadrilateral $BICI_A$ can be written as $\frac{m\sqrt{n}}{p}$, where $m$ and $p$ are relatively prime positive integers, and $n$ is a positive integer not divisible by the square of any prime. What is $m + n + p$?

**(A)** 31 **(B)** 32 **(C)** 33 **(D)** 34 **(E)** 35

25. Let $a$, $b$, $c$, and $d$ be real numbers with $a \geq b \geq c \geq d$ satisfying

$$\begin{aligned} a + b + c + d &= 0, \\ a^2 + b^2 + c^2 + d^2 &= 100, \\ a^3 + b^3 + c^3 + d^3 &= (a + b)(a + c)(b + c). \end{aligned}$$

The maximum possible value of $a^2 + b$ can be written as $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. What is $m + n$?

**(A)** 201 **(B)** 203 **(C)** 205 **(D)** 207 **(E)** 209

# SEASON 2 TMC 12

**DO NOT OPEN UNTIL THURSDAY, December 31, 2020**

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**Online Test Snowy Series**

*Questions and comments about problems and solutions for this exam should be sent by PM to:*

**Emathmaster** and **kevinmathz**.

The Season 2 OTIE will be held from January 8, 2021, to January 22, 2021. It is a 15-question, 3-hour, integer-answer exam. You will not be invited, but rather encouraged, to participate based on your score on this competition. *A complete listing of our publications may be found at our web site:*

`https://online-test-seasonal-series.github.io`

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****Administration On An Earlier Date Will Literally Be Impossible****

1. All the information needed to administer this exam is contained in the non-existent TMC 10/12 Teacher's Manual.
2. YOU must not verify on the non-existent TMC 10/12 COMPETITION CERTIFICATION FORM that you followed all rules associated with the administration of the exam.
3. Send DeToasty3, Emathmaster, jeteagle, and kevinmathz a PM submitting your answers to the TMC 12. AoPS is strongly recommended and is the only way to submit your answers.
4. The publication, reproduction, or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, Discord, Facebook, Hangouts or other digital media of any type during this period is a violation of the competition rules.

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*The* **Season 2 Online Snowy Tests**
*was made possible by the contributions of the following people:*

Aathreyakadambi, AIME12345, ARMLlegend, Awesome_guy, AwesomeYRY, azduncan, DeToasty3, Emathmaster, Ish_Sahh, ivyzheng, jeteagle, kevinmathz, MathCounts145, NJOY, Orestis_Lignos, PCChess, reaganchoi, rqhu, VulcanForge, & zhao_andrew

Finally, we thank you for taking this mock. We hope you enjoyed it!