Season 2 TMC 12 Solutions

Online Test Snowy Series

December 31, 2020, to January 13, 2021

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Answer Key:

1. (D)	2. (C)	3. (C)	4. (D)	5. (E)
6. (B)	7. (A)	8. (C)	9. (B)	10. (B)
11. (D)	12. (E)	13. (D)	14. (B)	15. (C)
16. (E)	17. (D)	18. (D)	19. (A)	20. (C)
21. (E)	22. (D)	23. (A)	24. (B)	25. (C)

Solutions:

1. In square $ABCD$, let M be the midpoint of side $\overline{CD}$, and let N be the reflection of M over side $\overline{AB}$. What fraction of $\triangle MND$ lies within $ABCD$?

   (A) $\frac{1}{2}$ **(B)** $\frac{5}{8}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{3}{4}$ **(E)** $\frac{7}{8}$

   Proposed by PCChess

   Answer (D): Let segment $\overline{MN}$ intersect side $\overline{AB}$ at point X and let segment $\overline{ND}$ intersect side $\overline{AB}$ at point Y. We have that triangles NYX and NDM are similar with ratio $\frac{1}{2}$. Thus, the area of NYX is $\frac{1}{4}$ of the area of NDM. The remaining part of NDM (besides NYX) lies within the square while triangle NYX does not. Thus, we have that $\boxed{\textbf{(D)}\ \frac{3}{4}}$ of its area lies within square $ABCD$. ■

2. Jack is reading a 100 page book. He reads two pages every minute. After every 12 pages he reads, he takes a one minute break, and then he goes back to reading. If Jack starts reading at 2:00, what time will it be when he finishes reading his book?

(A) 2:32 **(B)** 2:56 **(C)** 2:58 **(D)** 3:05 **(E)** 3:18

Proposed by ivyzheng

Answer (C): For every 12 full pages, we see that $\frac{12}{2}+1=7$ minutes pass. This will happen 8 times. The last $100-96=4$ pages will take $\frac{4}{2}=2$ minutes to read. Therefore, it will take Jack a total of $7\cdot 8+2=58$ minutes to finish the book. Adding this on to the time he starts reading, we get that he finishes reading at $\boxed{\textbf{(C) } 2{:}58}$. ■

3. Pedro currently has 2 quarters, 3 dimes and 2 pennies. If he can only obtain quarters, dimes, nickels, and pennies, what is the minimum number of coins he needs to earn in order to reach a total of exactly 1 dollar?

(A) 2 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 10

Proposed by PCChess

Answer (C): The amount of money that Pedro currently has (in cents) is $2\cdot 25+3\cdot 10+2\cdot 1=82$ cents. In order to reach exactly 1 dollar, he needs exactly 18 more cents. To achieve that with the minimum number of coins, we first need as many dimes as possible–there can only be at most 1 dime, which leaves us with 8 cents left. We can "fit" 1 nickel, and then 3 pennies. This gives us at least $1+1+3=\boxed{\textbf{(C) } 5}$ coins. ■

4. For how many integer values of b does the equation $2x^2+bx+5=0$ not have any real solutions?

(A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

Proposed by PCChess

Answer (C): Looking at the discriminant, we require that

$$b^2-40<0 \implies b^2<40.$$

This is only true when $|b|<\sqrt{40}$. Since b is an integer, we see that all of the integers from -6 to 6, inclusive, work, so the answer is $6-(-6)+1=\boxed{\textbf{(C) } 13}$. ■

5. Mark wants to distribute all 100 pieces of his candy to his five children, Albert, Bob, Charlie, Diana and Ethan. Diana and Ethan insist on each having a prime number of candies whose sum is also a prime number. Charlie insists on having exactly 35 candies, exactly 1 more than Albert and Bob's amounts combined. Given that Ethan has the smallest number of candies, how many candies must Mark give to Diana?

   (A) 3 **(B)** 7 **(C)** 17 **(D)** 23 **(E)** 29

   Proposed by ivyzheng

   Answer (E): If Diana and Ethan have a prime number of candies with the sum also being a prime number, then either Diana or Ethan must have 2 candies. Since it is given that Ethan has the smallest number of candies, we find that Ethan must have 2 candies. If Charlie has 1 more candy that Albert and Bob's amounts combined, then the sum of Albert and Bob's candies is 34. Since the total number of candies is 100, and the sum of Albert, Bob, Charlie, and Ethan's amounts is $34 + 35 + 2 = 81$, we have that the number of candies Mark gives to Diana is $\boxed{\textbf{(E)}\ 29}$, which we know is a prime number. ■

6. Positive real numbers a, b, c, d, and e satisfy $a + b + c + d + e = 2020$. Let S denote the sum of the minimum and maximum values of $\lfloor a \rfloor + \lfloor b \rfloor + \lfloor c \rfloor + \lfloor d \rfloor + \lfloor e \rfloor$, where $\lfloor r \rfloor$ denotes the largest integer less than or equal to r for all real numbers r. What is the sum of the digits of S?

   (A) 12 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16

   Proposed by ivyzheng

   Answer (B): Note that from the equation, we get that the sum of the integer parts and the fractional parts of a, b, c, d, e is 2020. We know that the fractional part of each number lies in the interval $[0, 1)$, so the sum of the fractional parts of a, b, c, d, e lies in the interval $[0, 5)$. Since 2020 is an integer, we know that this sum must be one of 0, 1, 2, 3, and 4. Thus, we have that the maximum value of $\lfloor a \rfloor + \lfloor b \rfloor + \lfloor c \rfloor + \lfloor d \rfloor + \lfloor e \rfloor$ is $2020 - 0 = 2020$, and the minimum value is $2020 - 4 = 2016$. A possible configuration for 2016 is when all of a, b, c, d, e have a fractional part of 0.8. Thus, we have that $S = 2020 + 2016 = 4036$, so the sum of the digits of S is $\boxed{\textbf{(B)}\ 13}$. ■

7. In a regular hexagon with side length 2, three of the sides are chosen at random. Next, the midpoints of each of the chosen sides are drawn. What is the probability that the triangle formed by the three midpoints has a perimeter which is an integer?

(A) $\frac{1}{10}$ **(B)** $\frac{1}{5}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{2}{5}$ **(E)** $\frac{1}{2}$

Proposed by DeToasty3

Answer (A): We see that there are three possible lengths of a side of the triangle: connecting two adjacent sides, connecting two sides one side apart, and connecting two opposite sides.

For the adjacent sides, we see that the length is one-half the length of the segment connecting two vertices with one vertex in between, which is $\frac{1}{2} \cdot 2\sqrt{3} = \sqrt{3}$, which is not an integer. For the sides one side apart, we see that the length is one-half the average of the side length and the length of the segment connecting two vertices with two vertices in between, which is $\frac{1}{2} \cdot (2+4) = 3$. Finally, for the two opposite sides, we see that the length is equal to the length of the segment connecting two vertices with one vertex in between, which is $2\sqrt{3}$.

Of these, we see that only the middle case has an integer length. There are two possible triangles with all three sides of this length. In total, there are $\binom{6}{3} = 20$ possible triangles, so our final probability is $\frac{2}{20} = \boxed{\textbf{(A)}\ \frac{1}{10}}$. ■

8. A positive integer is called a *flake* if it has at least three distinct prime factors. Two flakes are defined to be in a *snowflake* if there exists a prime that is divisible by the greatest common divisor of the two flakes. When two flakes in a snowflake are multiplied, what is the smallest possible number of divisors in the resulting number?

(A) 18 **(B)** 27 **(C)** 48 **(D)** 54 **(E)** 64

Proposed by Emathmaster

Answer (C): Let the first flake be $p_1^{e_1}p_2^{e_2}p_3^{e_3}$ (note that it is optimal to have less prime factors). Since the greatest common divisor (GCD) of the two flakes divides a prime, it is either a prime or 1. If the GCD is 1, then the other flake has 3 other distinct prime factors, all different from the prime factors of the original flake. This gives $(1+1)^6 = 64$ divisors of the product. If the GCD is a prime, then the two flakes share 1 prime factor. Thus, the number of divisors of the product is $(2+1)(1+1)^4 = 48$. The smallest possible number of divisors is $\boxed{\textbf{(C)}\ 48}$. ■

9. A group of people are in a room. It is given that 5 people have a pet dog, 6 people have a pet cat, 8 people have a pet fish, and 3 people have no pets. If no one has more than two pets, and no one has more than one of the same type of pet, what is the smallest possible number of people in the room?

(A) 10 **(B)** 13 **(C)** 14 **(D)** 16 **(E)** 19

Proposed by DeToasty3

Answer (B): To minimize the number of people in the room, we want to minimize the number of people who have at least one pet. In total, there are $5+6+8=19$ pets in the room. Thus, our minimum should be $\frac{19}{2}+\frac{1}{2}=10$ people with at least one pet. It suffices to find a construction. Let there be x people with both a pet dog and a pet cat, y people with both a pet cat and a pet fish, and z people with both a pet dog and a pet fish. Note that one person should have only a pet fish, so our 7 comes from $8-1$. Then, we have the following system of equations:

$$\begin{aligned} x+y&=6, \\ y+z&=7, \\ x+z&=5. \end{aligned}$$

From this, we get $x=2$, $y=4$, and $z=3$. This shows that the smallest number of people in the room is $10+3=\boxed{\textbf{(B)}\ 13}$, accounting for the three people who have no pets. ■

10. Let A and B be two distinct points in the plane. A ray is drawn emanating from A in any random direction, and a ray is drawn emanating from B in any random direction. What is the probability that the two rays intersect?

(A) $\frac{1}{8}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{3}{8}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{3}{4}$

Proposed by rqhu

Answer (B): Draw a line that passes through points A and B. WLOG, let B be to the right of A. Now, instead of considering rays emanating from points A and B, we consider lines passing through each of A and B. Note that these two lines are guaranteed to intersect, unless they are parallel or they are the same line, in which case the probability is 0 (these cases are negligible).

For each of the two lines, we can either choose the ray from A or B that goes above the line passing through A and B, or the ray that goes below it. Thus, there are $2^2=4$ possible choices for the two rays. Of these choices, only one choice results in the two rays intersecting, so the answer is $\boxed{\textbf{(B)}\ \frac{1}{4}}$. ■

11. For a positive integer n, define a function $f(n)$ to be equal to the largest integer k such that n is divisible by 2^k. For example, $f(8) = 3$ and $f(12) = 2$. Now, let p, q, and r be distinct primes less than 100, so that M is the largest value that

$$f((p+1)(q+1)(r+1))$$

can take, and m is the smallest value. What is $M + m$?

(A) 11 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16

Proposed by Emathmaster

Answer (D): Note that if we want to maximize $f((p+1)(q+1)(r+1))$, we will have to make p, q, r be one less than a number with a large exponent for 2. Since p, q, r are less than 100, the best that we can do is 64, a multiple of 32, a multiple of 16, and so on. Since $64 - 1 = 63$, which is not prime, we move on to check multiples of 32. We see that $32 - 1 = 31$, which is a prime, but $96 - 1 = 95$, which is not a prime (We have already checked 64, so we may skip it.). Next, we see that $48 - 1 = 47$, which is a prime, and $80 - 1 = 79$, which is a prime. We see that this is the best that we can do, where 32, 48, and 80 contribute 5, 4, and 4 to our count, respectively, giving $M = 5 + 4 + 4 = 13$.

Next, to minimize $f((p+1)(q+1)(r+1))$, we can let one of p, q, r be 2 and the other two be primes which are 1 modulo 4, e.g. 5 and 13. We see that this is the best that we can do, where 3, 6, and 14 contribute 0, 1, and 1 to our count, respectively, giving $m = 0 + 1 + 1 = 2$.

Thus, our answer is $M + m = 13 + 2 = \boxed{\textbf{(D) } 15}$. ■

12. A solution to $z^{27} - 1 = 0$ is chosen at random. What is the probability that it is also a solution to $\omega^6 + \omega^3 + 1 = 0$?

(A) $\frac{1}{27}$ **(B)** $\frac{1}{9}$ **(C)** $\frac{4}{27}$ **(D)** $\frac{5}{27}$ **(E)** $\frac{2}{9}$

Proposed by PCChess

Answer (E): The equation $z^{27} - 1 = 0$ has exactly 27 solutions, of the form $e^{\frac{2k\pi i}{27}}$ for $k = 0, 1, \ldots, 25, 26$. The solutions to $\omega^6 + \omega^3 + 1 = \frac{\omega^9 - 1}{\omega^3 - 1} = 0$ are of the form $e^{\frac{2k\pi}{9}}$ for $k = 0, 1, \ldots, 7, 8$, except for the solutions of the form $e^{\frac{2k\pi}{3}}$ for $k = 0, 1, 2$. Thus, the second equation has $9 - 3 = 6$ solutions. Checking denominators, we see that $9 \mid 27$, so all of the solutions to $\omega^6 + \omega^3 + 1 = 0$ are solutions to $z^{27} - 1 = 0$, so our desired probability is $\frac{6}{27} = \boxed{\textbf{(E) } \frac{2}{9}}$. ■

13. A sequence is defined recursively by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all integers $n \geq 2$. Let S be the sum of all positive integers n such that $|F_n - n^2| \leq 3n$. What is the sum of the digits of S?

(A) 6 **(B)** 7 **(C)** 10 **(D)** 11 **(E)** 14

Proposed by Emathmaster

Answer (D): We list out the first 14 values of F_n and n^2 (for $n = 1$ through $n = 14$). We find that values $1, 2, 3$ work. Then, 11 and 12 also work. After that, the Fibonacci numbers grow way faster than the perfect squares and thus there are no more solutions. Thus, $S = 1 + 2 + 3 + 11 + 12 = 29$ and thus the sum of its digits is $\boxed{\textbf{(D) } 11}$. ■

14. Let ABC be a triangle with circumcenter O and side lengths $AB = 9$, $BC = 7$, and $AC = 8$. If D is the reflection of A across line BO, then the length AD can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

(A) 25 **(B)** 43 **(C)** 44 **(D)** 50 **(E)** 57

Proposed by Ish_Sahh

Answer (B):

Solution 1, Law of Cosines (DeToasty3) Let $\angle BCA = \theta$. By the Law of Cosines, we have that

$$9^2 = 7^2 + 8^2 - 2(7)(8)\cos\theta \implies \cos\theta = \frac{2}{7}.$$

We see that $\angle BCA = \angle BDA$ because they intercept the same arc. Since D is the reflection of A over line BO, we see that $AB = BD = 9$. By Law of Cosines on $\triangle BDA$, we get that

$$9^2 = 9^2 + AD^2 - 2(9)(AD)\cos\theta \implies AD^2 - \frac{36}{7}AD = 0 \implies AD\left(AD - \frac{36}{7}\right) = 0.$$

Since $AD \neq 0$, we get that $AD = \frac{36}{7}$, so $m + n = \boxed{\textbf{(B) } 43}$. ■

Solution 2, coordinate bash (mathicorn) We place triangle ABC in the coordinate plane, with BC on the x-axis. Using Heron's formula, we find that the area of the triangle is $12\sqrt{5}$. We place point A on the y axis, and thus A has y coordinate $\frac{2 \cdot 12\sqrt{5}}{7} = \frac{24\sqrt{5}}{7}$. Since $AB = 9$, we know that the distance from point B to the origin is $\frac{33}{7}$ by the Pythagorean Theorem. Assume B is to the left of the origin and thus $B(\frac{-33}{7}, 0)$ and $C(\frac{16}{7}, 0)$.

We now need to find the coordinates of point O. Point O is the intersection of the 3 perpendicular bisectors of the triangle's 3 sides, but we can determine it using only 2 perpendicular bisectors. The perpendicular bisector of BC has equation $x = \frac{-17}{14}$. The perpendicular bisector of AC has equation $y = \frac{2\sqrt{5}}{15}x + \frac{164\sqrt{5}}{105}$. Point O is the intersection of these two lines, and we find the coordinates of O to be $(\frac{-17}{14}, \frac{7\sqrt{5}}{5})$.

Since D is the reflection of A across BO, AD is perpendicular to BO and the distance from A to BO is equal to the distance from D to BO, and both are equal to one half of AD. Thus, it remains to find the distance from point A to BO. The equation representing BO is $y = \frac{2\sqrt{5}}{5}x + \frac{66\sqrt{5}}{35}$.

Using distance from point to line formula, we get that the distance from A to BO is $\frac{18}{7}$ and thus $BD = 2 \cdot \frac{18}{7} = \frac{36}{7}$. Hence, $m + n = \boxed{\textbf{(B) } 43}$. ■

15. For how many positive integers $n \leq 15$ does there exist a positive integer k such that

$$\lfloor \log_2 k \rfloor + \lfloor \log_3 k \rfloor + \lfloor \log_4 k \rfloor + \cdots + \lfloor \log_8 k \rfloor = n?$$

(Here, $\lfloor r \rfloor$ denotes the largest integer less than or equal to r for all real numbers r.)

(A) 8 **(B)** 9 **(C)** 12 **(D)** 13 **(E)** 15

Proposed by DeToasty3

Answer (C): To solve this problem, we consider incrementing k until n exceeds 15. As we increment k, we see that the left hand side will increase when k is a perfect power; i.e. k can be expressed as a^b, where a and b are positive integers. For this problem, we also do not look at $a > 8$. Starting from $k = 1$, where the left hand side is 0, we see that the left hand side increases when $k = 2, 3, 4, 5, 6, 7, 8, 9, 16, 25, 27, 32 \ldots$ When $k = 32$, we see that the left hand side is equal to $5+3+2+2+1+1+1 = 15$. This means that once we reach the next perfect power, n will be greater than 15. The number of perfect powers between 1 and 32, inclusive, is 13. However, since n must be positive, we must exclude $k = 1$, which gives us an answer of $\boxed{\textbf{(C) } 12}$. ■

16. A fair coin is painted such that one side is red and the other side is blue. A fair die is painted such that all 6 faces are blue. Each move, Daniel flips the coin and rolls the die. He then paints the face facing up on the die the color of the side facing up on the coin. The probability that the die is completely red after 7 moves is $\frac{p}{12^q}$, where p and q are positive integers such that p is not divisible by 12. What is $p + q$?

(A) 35 **(B)** 75 **(C)** 110 **(D)** 180 **(E)** 215

Proposed by ivyzheng

Answer (E): We have that at most 1 move can be a repeat or a blue side painted blue. If a blue side is painted blue, we have 21 ways to choose the roll where this occurs. The other rolls are $\frac{5!}{6^5}$ chances and we multiply by $\frac{21}{6}$ to get $\frac{2520}{6^6}$ chance. Each occasion must follow a RB-pattern so we divide by 2^7. Now if we have seven red rolls, we have one space that's a repeat so we have $3 \cdot 7!$ ways to organize that with denominator $6^7 \cdot 2^7$. That gives $\frac{2520}{6^6}$ too, so we have $\frac{7!}{6^6 \cdot 2^7} = \frac{840}{6^5 \cdot 2^7} = \frac{210}{12^5}$. Our desired answer is $210 + 5 = \boxed{\textbf{(E) } 215}$. ■

17. How many distinct cubic polynomials $P(x)$ with all integer coefficients and leading coefficient 1 exist such that $P(0) = 3$, $|P(1)| < 12$, and $P(x)$ has three (not necessarily real or distinct) roots whose squares sum to 34?

$\textbf{(A) } 3 \qquad \textbf{(B) } 4 \qquad \textbf{(C) } 5 \qquad \textbf{(D) } 6 \qquad \textbf{(E) } 7$

Proposed by DeToasty3

Answer (D): Let r_1, r_2, and r_3 be the roots of the cubic polynomial. Observe that $r_1^2 + r_2^2 + r_3^2 = (r_1 + r_2 + r_3)^2 - 2(r_1r_2 + r_2r_3 + r_1r_3)$, so $34 = (r_1 + r_2 + r_3)^2 - 2(r_1r_2 + r_2r_3 + r_1r_3)$. With the cubic in the form $x^3 + bx^2 + cx + 3$, using Vieta's, we get $34 = b^2 - 2c$. We have that $|1 + b + c + 3| < 12$, so $-16 < b + c < 8$. Note that b and c are both integers, so we perform case work on b and c to find that (b, c) can take on ordered pairs $(-8, 15)$, $(-6, 1)$, $(-4, -9)$, $(2, -15)$, $(4, -9)$, and $(6, 1)$, for a total of $\boxed{\textbf{(D) } 6}$ polynomials. ■

18. A positive integer is called *edgy* if the sum of its digits divides the sum of the squares of its digits. For example, the number 19 is not an edgy number because $1 + 9 = 10$ does not divide $1^2 + 9^2 = 82$. How many two-digit edgy numbers are there?

$\textbf{(A) } 9 \qquad \textbf{(B) } 18 \qquad \textbf{(C) } 20 \qquad \textbf{(D) } 22 \qquad \textbf{(E) } 23$

Proposed by ARMLlegend

Answer (D): Let a be the tens digit and b be the ones digit of the number. Note that $a^2 + b^2 = (a + b)^2 - 2ab$, so $a + b \mid 2ab$. From here, we see that the number can be either a multiple of 10 or 11. We have 9 possibilities for a multiple of 10 and 9 possibilities for a multiple of 11. Now, we must check the greatest common divisor of a and b. Note that $\gcd(a, b) \in \{2, 3, 4\}$. We now check each case.

Case 1. $\gcd(a, b) = 2$. Then, let $a = 2a'$ and $b = 2b'$. Then, $a' + b' \mid 4a'b'$, and since a' and b' are relatively prime, we get that $a' + b' \mid 4$. Since we already counted the cases where $a' = b'$ in the multiples of 11 case, we have that the only cases here are $a' = 1$, $b' = 3$ and $a' = 3$, $b' = 1$, for a total of 2 additional cases.

Case 2. $\gcd(a, b) = 3$. Then, let $a = 3a'$ and $b = 3b'$. Then, $a' + b' \mid 9a'b'$, and since a' and b' are relatively prime, we get that $a' + b' \mid 9$. Since we already counted the cases where $a' = b'$ in the multiples of 11 case, we have that the only cases here are $a' = 1$, $b' = 2$ and $a' = 2$, $b' = 1$, for a total of 2 additional cases.

Case 3. $\gcd(a, b) = 4$. Then, let $a = 4a'$ and $b = 4b'$. Then, $a' + b' \mid 16a'b'$, and since a' and b' are relatively prime, we get that $a' + b' \mid 16$. Since we already counted the cases where $a' = b'$ in the multiples of 11 case, and every other case has already been covered in Case 1, we have 0 additional cases.

Thus, our total is $18 + 2 + 2 = \boxed{\textbf{(D)}\ 22}$. ■

19. A positive integer with $2n$ digits is *twisted* if the last n digits is some permutation of the first n digits, and the leading digit is nonzero. Let N be the number of twisted 6-digit integers. What is the sum of the digits of N?

(A) 18 **(B)** 21 **(C)** 24 **(D)** 27 **(E)** 30

Proposed by PCChess

Answer (A): We do casework based on the number of repeating digits in the first n digits.

Case 1. All 3 digits are the same. Then there are 9 ways to choose the first 3 digits. There are 9 numbers here.

Case 2. 2 digits are the same. We do casework again on if there is a zero.

- **Subcase 1.** If there is one 0, there are 9 ways to choose the other digit, and 2 ways to place the 0. Further, there are 3 ways to arrange the same digits in the last n place values, so there are $9 \cdot 2 \cdot 3 = 54$ numbers here.
- **Subcase 2.** If there are 2 zeros, there are 9 ways to choose the other digit, and 3 ways to arrange the digits in the second half. Our total here is $9 \cdot 3 = 27$.
- **Subcase 3.** If there are no zeros, there are 9 ways to choose the digit we use twice, 8 ways to choose the digit we use once, 3 ways to arrange the digits in the first n place values, and 3 ways to arrange the digits in the last n place values. Our total here is $9 \cdot 8 \cdot 3 \cdot 3 = 648$.

Our total here is $54 + 27 + 648 = 729$.

Case 3. All 3 digits are different. We do casework on if there is a zero.

- **Subcase 1.** If there is one zero, there are 2 ways to place the 0 and $9 \cdot 8$ ways to choose the other 2 digits. There are also 6 ways to arrange the digits in the latter half, so our total here is $9 \cdot 8 \cdot 2 \cdot 6 = 864$.
- **Subcase 2.** If there are no zeros, there are $9 \cdot 8 \cdot 7$ ways to choose the first n digits, and 6 ways to arrange them in the latter half. Our total here is $9 \cdot 8 \cdot 7 \cdot 6 = 3024$.

Our total here is $864 + 3024 = 3888$.

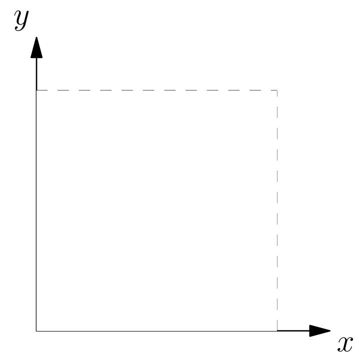
Summing, we get that $N = 9 + 729 + 3888 = 4626$, so our answer is $\boxed{\textbf{(A)}\ 18}$. ■

20. Andy and Aidan want to meet up at school to exchange phone numbers. Before meeting, they each choose a random time between 12:00 PM and 1:30 PM to arrive. After arriving, Andy will wait 40 minutes before leaving, while Aidan will wait 10 minutes before leaving. Later, Andy learns that he has an unexpected recital he has to attend at 1:10 PM, so if Andy is waiting for Aidan to arrive and the time passes 1:10 PM, Andy will leave, and if Andy chose a time after 1:10 PM, Andy will not come to the meeting at all. What is the probability that Andy and Aidan meet?

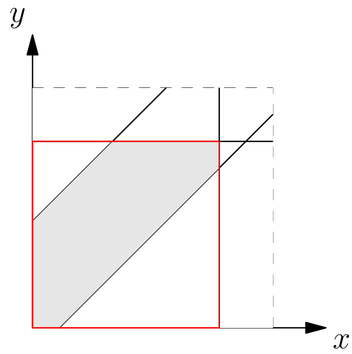
$\textbf{(A)}\ \frac{2}{9} \qquad \textbf{(B)}\ \frac{37}{162} \qquad \textbf{(C)}\ \frac{53}{162} \qquad \textbf{(D)}\ \frac{29}{81} \qquad \textbf{(E)}\ \frac{7}{18}$

Proposed by rqhu

Answer (C): Let the number of minutes after 12:00 PM that Andy arrives be x, and let the number of minutes after 12:00 PM that Aidan arrives be y. We set up a graph of what times are possible. We start with the axes and mark 0 and 90 and both axes (which represent the earliest and latest times they could arrive).



Suppose that Andy arrives before Aidan ($x < y$). Then Aidan must arrive in the next 40 minutes after Andy arrives, so $y - x \leq 40$. Similarly, if Aidan arrives before Andy, then Andy must arrive within 10 minutes of Aidan arriving, so $x - y \leq 10$. Finally, due to the unexpected meeting, the time that they meet must be before 1:10 PM, so $x, y < 70$. We add these inequalities to our diagram and mark the desired region:



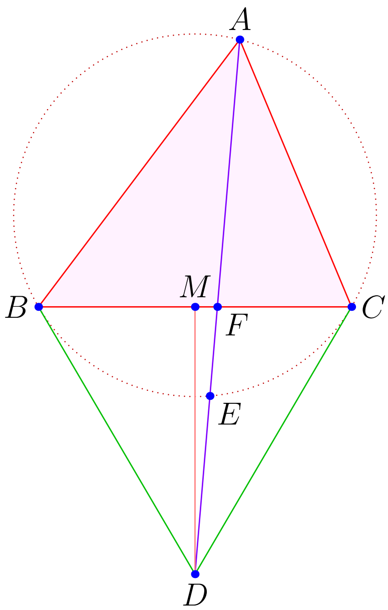
We find the area of this region by taking the area of the square outlined in red and then subtracting the areas of the right triangles in the upper left and bottom right. The area of the red square is $70^2 = 4900$. Since the slopes of the hypotenuses of the upper left and bottom right triangles are both 1, the triangles are isosceles right and the legs are equal. Note that the upper left triangle has its right angle vertex at $(0, 70)$ and one endpoint of the hypotenuse at $(0, 40)$. Thus, its leg length is $70 - 40 = 30$. Similarly, for the other triangle, one endpoint of the hypotenuse is at $(10, 0)$ and the right angle vertex is at $(70, 0)$. Thus, its leg length is $70 - 10 = 60$. Thus, the areas of the two triangles are $\frac{30^2}{2} = 450$ and $\frac{60^2}{2} = 1800$ respectively. The desired area is thus $4900 - 450 - 1800 = 2650$. The entire square's area is $90^2 = 8100$, so the desired probability is $\dfrac{2650}{8100} = \boxed{\textbf{(C)}\ \dfrac{53}{162}}$. ■

21. In acute $\triangle ABC$ with circumcircle Γ, the lines tangent to Γ at points B and C intersect at point D. Line segment $\overline{AD}$ intersects Γ at another point E, distinct from A, and $\overline{BC}$ at a point F. If $BF = 8$, $CF = 6$, and $EF = 4$, then the area of $\triangle BCD$ can be expressed as $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. What is $m + n$?

(A) 146 **(B)** 147 **(C)** 148 **(D)** 149 **(E)** 150

Proposed by Awesome_guy

Answer (E):



By Power of a Point, we have that $8 \cdot 6 = AF \cdot 4$, so $AF = 12$. Now, we wish to find DE. Note that $\triangle BCD$ is isosceles, so the foot of the perpendicular from D onto $\overline{BC}$ is its midpoint. Let M be the midpoint of $\overline{BC}$. Then, we have

$$BD^2 = DE(DE + 12 + 4)$$

by Power of a Point. Also, by the Pythagorean Theorem,

$$BD^2 = DM^2 + 7^2 = (DF^2 - FM^2) + 7^2.$$

Now, we see that $FM = BF - BM = 1$ and $DF = DE + 4$, so we obtain $DE = 8$, $DF = 12$, and $DM = \sqrt{143}$ after some simplifications. Finally, since $BC = 14$, we obtain that the area of $\triangle BCD$ is $\frac{1}{2} \cdot 14 \cdot \sqrt{143} = 7\sqrt{143}$. Thus, our final answer is $7 + 143 = \boxed{\textbf{(E) } 150}$. ■

22. For how many ordered triples of integers (a, b, c) between 1 and 10, inclusive, does

$$\frac{ab^2 + bc^2 + ca^2}{abc}$$

have a terminating (non-repeating) decimal expansion?

(A) 226 **(B)** 232 **(C)** 240 **(D)** 244 **(E)** 250

Proposed by Ish_Sahh

Answer (D): The idea is that if $x = \frac{a}{b}, y = \frac{b}{c}, z = \frac{c}{a}$, then

$$x + y + z = \frac{ab^2 + bc^2 + ca^2}{abc}.$$

Since $xyz = 1$, for all primes p, $\nu_p(x) + \nu_p(y) + \nu_p(z) = 0$.

Then, consider $\nu_p()$ of x, y, z. The decimal expansion will be terminating if $\nu_3(x+y+z) \geq 0$ and $\nu_7(x+y+z) \geq 0$.

Lemma 1: We either have $a = b = c = 7$, or $a, b, c \neq 7$. Assume that $\nu_7(x) > 0$ for some WLOG x. Then, $\nu_3(x + y + z) \leq \nu_7(x) + \nu_7(y) + \nu_7(z) = 0$'s equality case of $\nu_7(y) = \nu_7(z)$ is invoked. Thus, $\nu_7(x) = -\nu_7(y) - \nu_7(z) \leq -2$. Remember $x = \frac{a}{b}$, and $49 \nmid b$ so this is absurd. Thus, we either have $a = b = c = 7$, or $a, b, c \neq 7$.

Assume that $\nu_3(x) > 0$ for some WLOG x. Then, $\nu_3(x+y+z) \leq \nu_3(x)+\nu_3(y)+\nu_3(z) = 0$'s equality case of $\nu_3(y) = \nu_3(z)$ is invoked. Thus, $\nu_3(x) = -\nu_3(y) - \nu_3(z) \leq -2$. Remember $x = \frac{a}{b}$, so $b = 9$ and $3 \nmid a$.

We now count the number of triples. Call the 6 numbers $1, 2, 4, 5, 8, 10$ "nice numbers".

A triple of 3 nice numbers will always work so these 216 solutions automatically work.

The remaining triples must now consist of at least one number from $\{3, 6, 7, 9\}$. $(3, 3, 3), (6, 6, 6), (7, 7, 7), (9, 9, 9)$ gives 4 solutions. Moreover, any further solutions will not contain 7 by Lemma 1. Thus, we must have at least one number from $\{3, 6, 9\}$. If the triple does not contain 9, then the permutations of $(3, 3, 6)$ and $(3, 6, 6)$ give a total of 6 solutions. If the triple does contain 9, then $a = 9, b = 2, 5, 8, c = 3, 6$ and their rotations give 18 solutions.

This gives a total of $216 + 4 + 6 + 18 = \boxed{\textbf{(D) } 244}$ solutions. ■

23. Consider the quadratic equation $P(x) = ax^2 + bx + 144$, where a and b are real numbers. It is known that $P(x)$ has two distinct positive integer roots, and its graph is tangent to the graph of $y = -x^2$. The sum of all possible values of $\frac{1}{a}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

(A) 61 **(B)** 62 **(C)** 63 **(D)** 64 **(E)** 65

Proposed by reaganchoi

Answer (A): The key is to reverse the quadratic into the form $144x^2+bx+a$ (although other methods probably work too). This must have two unit-fraction roots, $\frac{1}{r}$ and $\frac{1}{s}$. In addition, since the quadratic $(a+1)x^2+bx+144$ has exactly one real root, so must $144x^2+bx+(a+1)$; in other words, our quadratic of $144x^2+bx+a$ has a vertex with y-coordinate -1.

The roots are $\frac{1}{r}$ and $\frac{1}{s}$; WLOG, let $r > s$ so $\frac{1}{s} > \frac{1}{r}$. Then, the vertex must be at $\left(\frac{\frac{1}{r}+\frac{1}{s}}{2}, -1\right)$. Writing this in vertex form:

$$144\left(x - \frac{\frac{1}{r}+\frac{1}{s}}{2}\right)^2 - 1$$

Note that plugging in $x = \frac{1}{r}$ gives 0. Thus, we have

$$144\left(\frac{\frac{1}{s}-\frac{1}{r}}{2}\right)^2 - 1 = 0.$$

$$36\left(\frac{1}{s}-\frac{1}{r}\right)^2 = 1.$$

$$\frac{1}{s}-\frac{1}{r} = \frac{1}{6}.$$

Then, we can expand this to get $6r - 6s = rs$, so $rs - 6r + 6s = 0$. Solving this Diophantine equation gives $(r+6)(s-6) = -36$; since $r, s > 0$, we need $r + 6 > 6$, so our only possible product pairs are $(9, -4)$, $(12, -3)$, $(18, -2)$, and $(36, -1)$, giving pairs $(r, s) = (3, 2)$, $(6, 3)$, $(12, 4)$, and $(30, 5)$.

Note that, by Vieta's, $\frac{144}{a} = rs$. Thus, to find the sum of all $\frac{1}{a}$, we need to find the sum of all $\frac{rs}{144}$. This sum can be evaluated to be $\frac{6+18+48+150}{144} = \frac{222}{144} = \frac{37}{24}$, so the answer is $37 + 24 = \boxed{\textbf{(A) } 61}$. ■

24. Let $\triangle ABC$ have $\angle BAC = 60°$ and incenter I. Let ω_A be a circle in the exterior of $\triangle ABC$ that is tangent to side $\overline{BC}$ and the extensions of the other two sides. Let I_A be the center of ω_A. If $BI_A = 4$ and $CI_A = 3$, then the area of quadrilateral $BICI_A$ can be written as $\frac{m\sqrt{n}}{p}$, where m and p are relatively prime positive integers, and n is a positive integer not divisible by the square of any prime. What is $m + n + p$?

(A) 31 **(B)** 32 **(C)** 33 **(D)** 34 **(E)** 35

Proposed by Emathmaster

Answer (B): The incenter-excenter lemma states that $BICI_A$ is cyclic with diameter $\overline{II_A}$. Explicitly, $IC \perp CI_A$ and $IB \perp BI_A$.

Note that

$$\angle BIC = 180° - \frac{\angle B}{2} - \frac{\angle C}{2} = 90° + \frac{\angle A}{2} = 90° + 30° = 120°.$$

Thus, $\angle BI_AC = 180° - 120° = 60°$.

By Law of Cosines,

$$BC^2 = I_AB^2 + I_AC^2 - 2\cos(60^\circ) \cdot I_AB \cdot I_AC = 3^2 + 4^2 - 2 \cdot 1/2 \cdot 3 \cdot 4 = 13.$$

By Law of Sines, the diameter of $BICI_A$ has length $\frac{BC}{\sin(60^\circ)} = \frac{2\sqrt{13}}{\sqrt{3}} = \sqrt{\frac{52}{3}}$. Then, by the Pythagorean Theorem,

$$IC = \sqrt{\frac{52}{3} - 3^2} = \frac{5\sqrt{3}}{3},$$
$$IB = \sqrt{\frac{52}{3} - 4^2} = \frac{2\sqrt{3}}{3}.$$

Finally, the area of $BICI_A$ is

$$\frac{BI \cdot BE + CI \cdot CE}{2} = \frac{\frac{2\sqrt{3}}{3} \cdot 4 + \frac{5\sqrt{3}}{3} \cdot 3}{2} = \frac{23\sqrt{3}}{6}.$$

Our answer is $m + n + p = 23 + 3 + 6 = \boxed{\textbf{(B) } 32}$. ■

25. Let a, b, c, and d be real numbers with $a \geq b \geq c \geq d$ satisfying

$$a + b + c + d = 0,$$
$$a^2 + b^2 + c^2 + d^2 = 100,$$
$$a^3 + b^3 + c^3 + d^3 = (a + b)(a + c)(b + c).$$

The maximum possible value of $a^2 + b$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

(A) 201 **(B)** 203 **(C)** 205 **(D)** 207 **(E)** 209

Proposed by ARMLlegend

Answer (C):

Solution 1 (ARMLlegend) We can rewrite the last equation as

$$(b^3 + c^3) + (a^3 + d^3) = (a + b)(a + c)(b + c)$$
$$\implies (b + c)(b^2 - bc + c^2) + (a + d)(a^2 - ad + d^2) = (a + b)(a + c)(b + c).$$

For the sake of contradiction, let $b + c \neq 0$. Divide both sides to get

$$b^2 - bc + c^2 - (a^2 - ad + d^2) = (a + b)(a + c) = a(a + b + c) + bc = bc - ad,$$

so now we have

$$b^2 - bc + c^2 = a^2 - ad + d^2 + bc - ad \implies (b-c)^2 = (a-d)^2,$$

and since both $b-c$ and $a-d$ are nonnegative, we have

$$b - c = a - d \implies a + c = b + d = 0.$$

Since $a \geq b \geq c \geq d$, this yields $a = b$ and $c = d$, but this contradicts $b + c \neq 0$. Thus, we must have $b + c = a + d = 0$. From the second equation we have $a^2 + b^2 = 50$, so $a^2 + b = -b^2 + b + 50$, which, at $b = \frac{1}{2}$, has a maximum of $\frac{201}{4}$. It can be easily verified that the quadruple $(a, b, c, d) \equiv \left(\frac{\sqrt{199}}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{\sqrt{199}}{2}\right)$ indeed satisfies the equation. Hence, the maximum $\frac{201}{4}$ is possible, so, $m + n = \boxed{\textbf{(C) } 205}$. ■

Solution 2 (DeToasty3) It is well known that

$$(a+b+c)^3 = a^3 + b^3 + c^3 + 3(a+b)(a+c)(b+c)$$

$$\implies a^3 + b^3 + c^3 = (a+b+c)^3 - 3(a+b)(a+c)(b+c),$$

so the third equation becomes

$$(a+b+c)^3 - 3(a+b)(a+c)(b+c) + d^3 = (a+b)(a+c)(b+c).$$

From the first equation, we get

$$(-d)^3 + d^3 = 4(a+b)(a+c)(b+c) \implies (a+b)(a+c)(b+c) = 0.$$

Now, one of $a+b$, $a+c$, and $b+c$ is equal to 0. From the first equation, we must have that $a = -d$ and $b = -c$ to satisfy $a \geq b \geq c \geq d$. From the second equation, we get that $a^2 + b^2 = 50$.

Adding $b - b^2$ to both sides of the equation $a^2 + b^2 = 50$, we get $a^2 + b = -b^2 + b + 50$. We see that the right hand side is equal to $-\left(b - \frac{1}{2}\right)^2 + \frac{201}{4}$, so by letting $b = \frac{1}{2}$, we get that the maximum possible value of $a^2 + b$ is $\frac{201}{4}$, so $m + n = \boxed{\textbf{(C) } 205}$. ■