

# Oi, Try Judging a Math Olympiad — Day

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OTJMO, 2021 ★ *Online Everywhere* ★★ *4 hours and 30 minutes* ★ *7 points per problem*

**1.** Call an ordered pair of real numbers  $(a, b)$  *good* if the quadratics  $x^2 + ax + b$  and  $x^2 + bx + a$  both are greater than or equal to  $k$  for all real  $x$ , where  $k$  is a real number. Find all values of  $k$  such that there are at least two good pairs of real numbers  $(a, b)$ .

**2.** Let  $n$  be an odd positive integer. All positive integers less than or equal to  $n$  are written on the board. Alan performs the following move:

He picks two different numbers from the board that are both even or both odd, removes them and writes their average twice instead. For example, if he picks  $(4, 6)$ , he will then write  $(5, 5)$  instead.

Prove that Alan can perform only finitely many moves, and for each  $n$ , find the maximum number of moves he can perform.

**3.** Pat has a device that marks all points  $X$  on the line  $P_1P_2$  such that  $\frac{P_1X}{P_2X} = \frac{Q_1Q_2}{R_1R_2}$  for six given points  $P_1 \neq P_2$ ,  $Q_1 \neq Q_2$  and  $R_1 \neq R_2$ . Prove that given any three non-collinear points  $A, B, C$ , Pat can mark the orthocenter of  $ABC$  using only this device.