Oi, Try Judging a Math Olympiad — Day

OTJMO, 2021 ★ *Online Everywhere* ★★ *4 hours and 30 minutes* ★ *7 points per problem*

1. Call an ordered pair of real numbers (a, b) *good* if the quadratics $x^2 + ax + b$ and $x^2 + bx + a$ both are greater than or equal to k for all real x, where k is a real number. Find all values of k such that there are at least two good pairs of real numbers (a, b).

2. Let n be an odd positive integer. All positive integers less than or equal to n are written on the board. Alan performs the following move:

> He picks two different numbers from the board that are both even or both odd, removes them and writes their average twice instead. For example, if he picks $(4, 6)$, he will then write $(5, 5)$ instead.

Prove that Alan can perform only finitely many moves, and for each n, find the maximum number of moves he can perform.

3. Pat has a device that marks all points X on the line P_1P_2 such that $\frac{P_1X}{P_2X} = \frac{Q_1Q_2}{R_1R_2}$ for six given points $P_1 \neq P_2$, $Q_1 \neq Q_2$ and $R_1 \neq R_2$. Prove that given any three non-collinear points A, B, C, Pat can mark the orthocenter of ABC using only this device.

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4. Call a positive integer m *cool* if it can be expressed as $7^x - 9^y$ for some positive integers x and y. Can the product or the sum of two cool integers ever be cool?

5. Triangle ABC, right angled at A, has circumcircle Γ. Point D on arc $\widehat{AB}$ and point E on arc $\widehat{AC}$ of Γ lie such that $AD = AE$. Lines BD and CE meet at K. The tangents to Γ at D and E meet at T. If the circumcircle of $\triangle DKT$ meets Γ again at M, and lines AB and KT meet at N, prove that the circumcenter of $\triangle BMN$ lies on KT.

6. Ivy reaches a magical world where she finds an infinite number of gift boxes, each having a different number of *bitcoins* inside them (i.e. at most a single box may be empty). She can choose k boxes, and she will receive all the bitcoins present in all those k boxes. Before she begins, she can randomly peek into k of the boxes and count the number of bitcoins in each of them. If Ivy can add properly, what maximum number of bitcoins (in terms of k) is she guaranteed to receive?