Oi, Try Judging a Math Olympiad
Report — OTJMO 2021

OTSS Committee

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0 Problems

0.1 Day 1 Problems

1. Call an ordered pair of real numbers (a, b) *good* if the quadratics $x^2 + ax + b$ and $x^2 + bx + a$ both are greater than or equal to k for all real x, where k is a real number. Find all values of k such that there are at least two good pairs of real numbers (a, b).

2. Let n be an odd positive integer. All positive integers less than or equal to n are written on the board. Alan performs the following move:

> He picks two different numbers from the board that are both even or both odd, removes them and writes their average twice instead. For example, if he picks $(4, 6)$, he will then write $(5, 5)$ instead.

Prove that Alan can perform only finitely many moves, and for each n, find the maximum number of moves he can perform.

3. Pat has a device that marks all points X on the line P_1P_2 such that $\frac{P_1X}{P_2X} = \frac{Q_1Q_2}{R_1R_2}$ for six given points $P_1 \neq P_2$, $Q_1 \neq Q_2$ and $R_1 \neq R_2$. Prove that given any three non-collinear points A, B, C, Pat can mark the orthocenter of ABC using only this device.

0.2 Day 2 Problems

4. Call a positive integer m *cool* if it can be expressed as $7^x - 9^y$ for some positive integers x and y. Can the product or the sum of two cool integers ever be cool?

5. Triangle ABC, right angled at A, has circumcircle Γ. Point D on arc $\widehat{AB}$ and point E on arc $\widehat{AC}$ of Γ lie such that $AD = AE$. Lines BD and CE meet at K. The tangents to Γ at D and E meet at T. If the circumcircle of $\triangle DKT$ meets Γ again at M, and lines AB and KT meet at N, prove that the circumcenter of $\triangle BMN$ lies on KT.

6. Ivy reaches a magical world where she finds an infinite number of gift boxes, each having a different number of *bitcoins* inside them (i.e. at most a single box may be empty). She can choose k boxes, and she will receive all the bitcoins present in all those k boxes. Before she begins, she can randomly peek into k of the boxes and count the number of bitcoins in each of them. If Ivy can add properly, what maximum number of bitcoins (in terms of k) is she guaranteed to receive?

1 Day 1 Solutions

1.1 Problem 1, by Orestis_Lignos

Problem 1. Call an ordered pair of real numbers (a, b) *good* if the quadratics $x^2 + ax + b$ and $x^2 + bx + a$ both are greater than or equal to k for all real x, where k is a real number. Find all values of k such that there are at least two good pairs of real numbers (a, b).

Answer — The answer is all $k < 1$.

Solution. Let (a, b) be a good pair, then $x^2 + ax + b \geq k$ for all x, so the discriminant is ≤ 0, i.e. $a^2 \leq 4(b-k)$ and similarly $b^2 \leq 4(a-k)$.

Thus,

$$8k = 4k + 4k \leq 4b - a^2 + 4a - b^2 = (4a - a^2) + (4b - b^2) \leq 4 + 4 = 8,$$

where the inequalities follow from AM-GM.

Therefore, $k \leq 1$. If $k = 1$ we have equality everywhere, so there is only one good pair, $(2, 2)$, a contradiction.

Now we'll prove that all $k < 1$ yield at least two good pairs. Let $a = 2$, then we need $4(b-k) \geq 4$, i.e. $b \geq k+1$, and $b^2 \leq 4(2-k)$, i.e. $b \leq \sqrt{8-4k}$.

We will prove that for all $k < 1$, $k + 1 < \sqrt{8-4k}$ holds true, yielding infinitely many good pairs.

If $k \leq -1$ we are trivially done, so suppose $k > -1$. Then, taking the square we have to prove $(k+1)^2 + 4k \leq 8$, that is $(k-1)(k+7) \leq 0$, which is true since $k - 1 \leq 0$ and $k + 7 > -1 + 7 = 6 > 0$. $\square$

1.2 Problem 2, by Orestis_Lignos

Problem 2. Let n be an odd positive integer. All positive integers less than or equal to n are written on the board. Alan performs the following move:

> He picks two different numbers from the board that are both even or both odd, removes them and writes their average twice instead. For example, if he picks $(4, 6)$, he will then write $(5, 5)$ instead.

Prove that Alan can perform only finitely many moves, and for each n, find the maximum number of moves he can perform.

Solution. Define $S = a_1^2 + a_2^2 + \cdots + a_n^2$ where a_i are the numbers on the board at some point.

Then, if we pick a, b and write $\dfrac{a+b}{2}, \dfrac{a+b}{2}$ instead, the new value of S will be

$$S' = S - a^2 - b^2 + \frac{(a+b)^2}{2} = S - \frac{(a-b)^2}{2},$$

and since $a \neq b$, we must have $S' < S$, hence S constantly decreases. Thus, if Alan could perform infinitely many moves we would get a contradiction, since $S > 0$. So, he can only perform finitely many moves.

We have $S' = S - \dfrac{(a-b)^2}{2}$, and since $a - b \equiv 0 \pmod 2, a \neq b$, we have $|a-b| \geq 2$, whence $(a-b)^2 \geq 4$, therefore $S' \leq S - 2$, so S decreases by at least 2 ay each move. Suppose that m is the maximum number of moves we can perform.

Then, if t_i are the n numbers after the m-th move, we would have

$$\sum t_i^2 \leq \sum i^2 - 2m,$$

where both sums run from 1 to n.

In addition, note that the move $a, b \to \frac{a+b}{2}, \frac{a+b}{2}$ doesn't change the total sum of the numbers in the board, hence we must have $\sum t_i = 1 + 2 + \ldots n = \dfrac{n(n+1)}{2}$

Thus, by Cauchy-Schwarz, $\sum t_i^2 \geq \dfrac{(\sum t_i)^2}{n} = \dfrac{n(n+1)^2}{4}$, therefore

$$\sum i^2 - 2m \geq \frac{n(n+1)^2}{4} \Rightarrow \frac{n(n+1)(2n+1)}{6} - 2m \geq \frac{n(n+1)^2}{4} \Rightarrow m \leq \frac{n^3-n}{24}.$$

Note that, since n is odd, $\frac{n^3-n}{24}$ is an integer because $8 \mid (n^2-1)$ and $3 \mid n(n-1)(n+1)$. Now, we have to prove that we can indeed make $\frac{n^3-n}{24}$ moves.

In order for this to happen, equality must hold at Cauchy-Schwarz. In addition, $|a-b|$ must equal 2 for each two numbers a, b we pick.

Hence, after the m-th move we should have obtained the position with n numbers equal to $\frac{\frac{n(n+1)}{2}}{n} = \frac{n+1}{2}$, and we only perform moves where the two numbers we pick differ by 2.

We will consider these two claims:

Claim — We can change $a, a+1, \dots, a+2k$ to $a+1, a+2, \dots, a+(2k-1), a+1, a+(2k-1)$ for any $k \geq 1$.

Proof. This is possible by doing the move $a+i, a+i+2 \to a+i+1, a+i+1$ for $i = 0, 1, \dots, 2k-2$ in that order. We'll call this set of moves $P(a, k)$. ■

Claim — We can change $a, a+1, \dots, a+2k$ to $a+1, \dots, a+(2k-1), a+k, a+k$ for any $k \geq 1$.

Proof. This is possible by doing the set of moves $P(a+i, k-i)$ for $i = 0, 1, \dots, k-1$ in that order. We'll call this set of moves $Q(a, k)$. ■

Now, we are done by using the set of moves $Q(a+i, k-i)$ for $i = 0, 1, \dots, k-1$ in that order. □

1.3 Problem 3, by Mindstormer

Problem 3. Pat has a device that marks all points X on the line P_1P_2 such that $\frac{P_1X}{P_2X} = \frac{Q_1Q_2}{R_1R_2}$ for six given points $P_1 \neq P_2$, $Q_1 \neq Q_2$ and $R_1 \neq R_2$. Prove that given any three non-collinear points A, B, C, Pat can mark the orthocenter of ABC using only this device.

Solution. We break the problem into several claims, which essentially are the steps presented with their justifications.

Claim — We can mark midpoints of any segments using the device.

Proof. Choose $Q_1 = R_1$, $Q_2 = R_2$. This makes $P_1X/P_2X = 1$. The claim follows. ■

Also, in any triangle XYZ we can mark the feet of internal and external angle bisectors by dividing, say, XY in the ratio $\frac{ZX}{ZY}$.

We will first prove that we can mark H_a, the foot of the altitude from A onto BC. If $AB = AC$, it's just the midpoint of BC, which we can mark. Otherwise, let AL and AK be internal and external bisectors of $\angle A$ in $\triangle ABC$. Again, if $AK = AL$, we just mark the midpoint of KL. Otherwise, let AP and AQ be the internal and external bisectors of $\angle A$ in $\triangle AKL$. If S is the midpoint of PQ, then S is the center of the Apollonius circle of $\triangle AKL$, so $\frac{KS}{LS} = \frac{AK^2}{AL^2}$. Since AKL is the altitude in right-angled triangle AKL, we have $\frac{H_aK}{H_aL} = \frac{AK^2}{AL^2} = \frac{KS}{LS}$, so we can mark H_a.

Similarly, we can mark H_b and H_c, the feet of altitudes from B and C. Let H_aN be an internal angle bisector in $\triangle H_aH_bH_c$ and H_bH be an internal angle bisector in $\triangle H_aH_bN$. Then H is the incenter of $\triangle H_aH_bH_c$, which is the orthocenter of ABC, as desired. □

2 Day 2 Solutions

2.1 Problem 4, by Orestis_Lignos

Problem 4. Call a positive integer m *cool* if it can be expressed as $7^x - 9^y$ for some positive integers x and y. Can the product or the sum of two cool integers ever be cool?

Solution. The solution naturally divides into two parts:

Part 1: The product of two cool numbers is never cool.

Suppose there existed cool numbers p, q, r such that $pq = r$. Let $p = 7^x - 9^y, q = 7^z - 9^w, r = 7^t - 9^s$, then

$$(7^x - 9^y)(7^z - 9^w) = 7^t - 9^s$$

Taking $\pmod{7^{\min\{x,z,t\}}}$ we obtain that $7^{\min\{x,z,t\}} \mid 9^{y+w} + 9^s$, hence $7 \mid u^2 + v^2$ with $u = 3^{y+w}$, $v = 3^s$, which is clearly impossible by Fermat's Christmas Theorem (or just check quadratic residues $\pmod 7$)

Part 2: The sum of two cool numbers is never cool

Suppose there existed cool numbers p, q, r such that $p + q = r$. Let $p = 7^x - 9^y, q = 7^z - 9^w, r = 7^t - 9^s$, then

$$7^x - 9^y + 7^z - 9^w = 7^t - 9^s,$$

hence again taking $\pmod{7^{\min\{x,z,t\}}}$ we obtain that $7 \mid 9^y + 9^w - 9^s$.

We can easily prove that $9^\delta \equiv 1, 2 \text{ or } 4 \pmod 7$ by taking $\delta = 3\epsilon + \upsilon$ with $\upsilon\{0, 1, 2\}$.

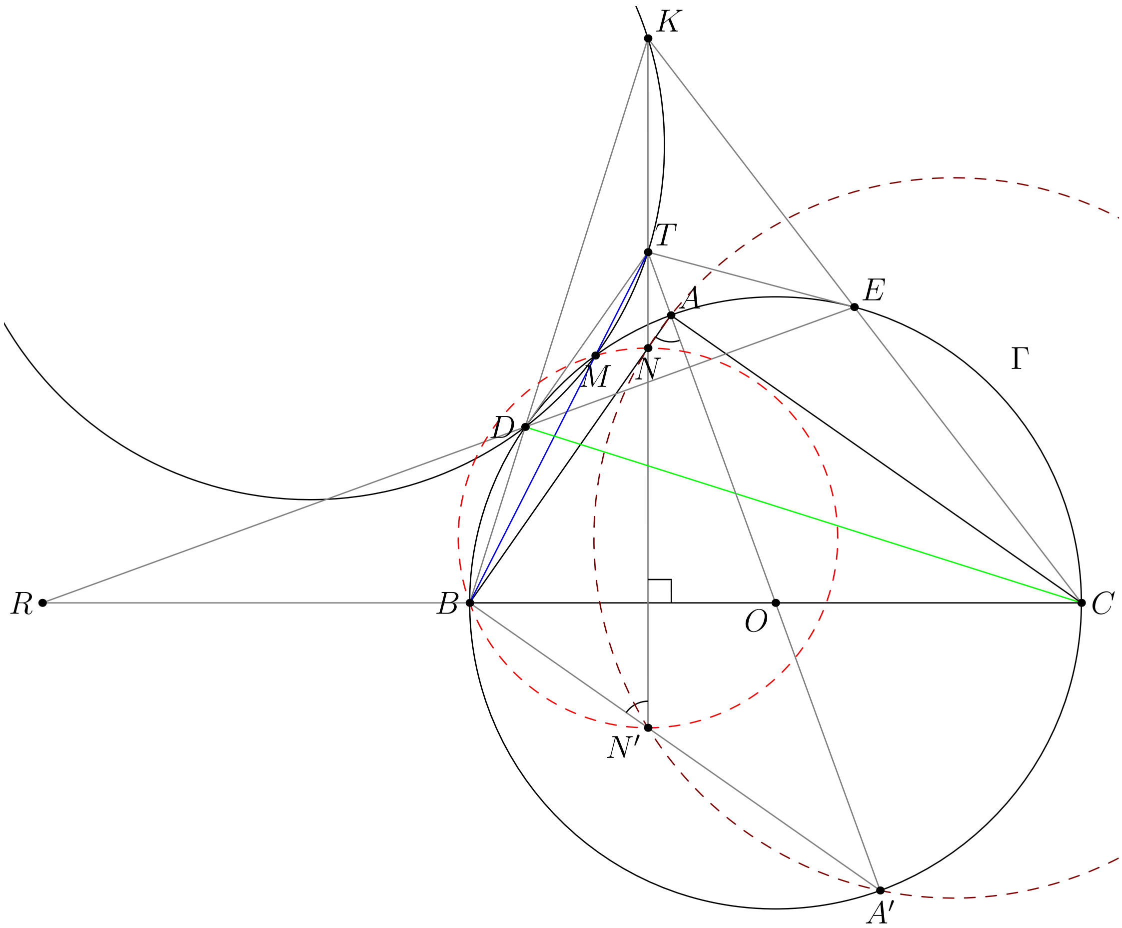
Now some easy case bash proves that $7 \mid 9^y + 9^w - 9^s$ can never hold, hence we are done. $\square$

2.2 Problem 5, by Awesome_guy

Problem 5. Triangle ABC, right angled at A, has circumcircle Γ. Point D on arc $\widehat{AB}$ and point E on arc $\widehat{AC}$ of Γ lie such that $AD = AE$. Lines BD and CE meet at K. The tangents to Γ at D and E meet at T. If the circumcircle of $\triangle DKT$ meets Γ again at M, and lines AB and KT meet at N, prove that the circumcenter of $\triangle BMN$ lies on KT.

⚠ $\measuredangle$ refers to directed angles, as more than one configurations are possible.

Solution. Let DE meet BC at point R. By Brocard's Theorem, line KT is the pole of point R. But, since line BC passes through the center of Γ (call O), we obtain $KT \perp BC$.



Claim — The points B, M and T are collinear.

Proof. As a result of $KT \perp BC$, we obtain

$$\measuredangle DMT = \measuredangle DKT = (90^\circ - \measuredangle CBD)$$
$$= \measuredangle DCB = \measuredangle DMB,$$

which implies that points B, M, T are collinear. ■

Let A' be the *antipode* of A in Γ. Let KT meet BA' at point N'. Now, observe that $\measuredangle NBN' = \measuredangle ABA' = 90^\circ$.

It just leaves us to show that the points B, M, N, N' are concyclic, and so, their center will be midpoint of NN' which lies on line KT. We prove another claim to proceed.

Claim — The points A, N, N', A' are concyclic.

Proof. Note that

$$\begin{aligned} \measuredangle NAA' &= \measuredangle BAA' = \measuredangle CBA \\ &= 90^\circ - \measuredangle A'BC = \measuredangle NN'B \\ &= \measuredangle NN'A', \end{aligned}$$

which implies that N, N', A', A are concyclic. ■

Claim — Points T, A, A' are collinear.

Proof. Observe that T is the intersection point of tangents at D and E to Γ, and so, T lies on perpendicular bisector of DE. By the problem statement, A too lies on it and by definition, O will also lie on the perpendicular bisector of DE. Hence, T, A, O and as a consequence, T, A, A' are collinear. ■

To finish, by Power of Point at T, we obtain

$$TB \cdot TM = TA \cdot TA' = TN \cdot TN'$$

which yields points B, M, N, N' as concyclic. Hence, the conclusion follows. □

2.3 Problem 6, by TLP.39

Problem 6. Ivy reaches a magical world where she finds an infinite number of gift boxes, each having a different number of *bitcoins* inside them (i.e. at most a single box may be empty). She can choose k boxes, and she will receive all the bitcoins present in all those k boxes. Before she begins, she can randomly peek into k of the boxes and count the number of bitcoins in each of them. If Ivy can add properly, what maximum number of bitcoins (in terms of k) is she guaranteed to receive?

Answer — The answer is $x = k^2 - \left\lfloor \frac{k}{2} \right\rfloor$.

Solution. We proceed to prove the bound in two steps.

Proof that she can get at least this amount of bitcoins:

If the number of bitcoins in those k boxes that she has already opened is already at least x, then just choose those k boxes.

Else, randomly choose k boxes that aren't opened yet. Since all boxes have different amount of bitcoins, the total number of bitcoins in the k boxes we choose and k boxes we looked into will be at least $0 + 1 + \cdots + (2k - 1) = 2k^2 - k$ this means that we get at least $2k^2 - k - (x - 1) = k^2 - \left\lceil \frac{k}{2} \right\rceil + 1 \geq x$ bitcoins.

So, both cases imply she is guaranteed to receive at least x bitcoins.

Proof that she cannot guarantee to get $x + 1$ bitcoins or more:

If k is even and is equal to $2h$, consider the case where, upon opening k boxes, she found out that the k boxes have $\{h, h+1, \ldots, 3h-1\}$ bitcoins. If she decide to choose $m \leq h$ of the boxes she looked into, then she can get at most $\left(\sum_{i=3h-m}^{3h-1} i\right)$ bitcoins from them, and for the rest $k - m$ boxes, the worst case is that she get

$$\sum_{i=0}^{h-1} i + \sum_{i=3h}^{4h-m-1} i$$

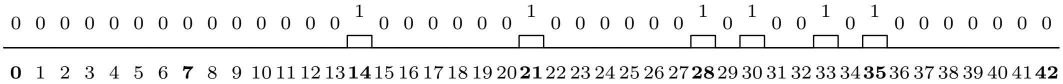
bitcoins. In total, she can only guarantee $4h^2 - (1+m)h \leq 4h^2 - h = x$ bitcoins. Else if she chooses $m > h$ of the boxes that she looked into, she can guarantee at most $\left(\sum_{i=3h-m}^{3h-1} i\right)$ bitcoins from those m boxes and at most $\left(\sum_{i=0}^{k-m-1} i\right)$ bitcoins from the rest, with total of $2h^2 - h + hm \leq 4h^2 - h = x$ bitcoins.

If k is odd and is equal to $2h+1$, then consider the case where she found $\{h, h+1, \ldots, 3h-1\}$ bitcoins for the first $k-1$ boxes and $4h$ for the last. If she chooses those k boxes, she gets $x - 1$ bitcoins. Otherwise, if she doesn't choose any of them, she can only guarantee to get x bitcoins in the worst case (getting $\{0, 1, \ldots, h-1, 3h, 3h+1, \ldots, 4h-1, 4h+1\}$ bitcoins). Else, this is similar to previous case, but with $4h$ more bitcoins from the boxes with highest number of bitcoins that she looked into. Thus she can only guarantee $4h^2 - h + 4h = x - 1$ bitcoins. This also does not add to x bitcoins.

Hence, we have exhausted all the cases, and so, the maximum number of bitcoins she can receive is as mentioned in the answer. □

3 Results

3.1 Score Summary



3.2 Leaderboard

Username	P1	P2	P3	P4	P5	P6	Score
primesarespecial	7	7	7	7	7	-	35
v4913	7	1	4	7	7	7	33
CrazyInMath	7	1	2	7	7	6	30
sanyalarnab	7	7	7	7	-	0	28
asdf334	7	-	-	7	7	-	21
math31415926535	0	-	-	7	7	0	14

4 Acknowledgements

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